Advanced Extremal Models for Operational Risk

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1 Introduction

Managing risk lies at the heart of the financial services industry. Regulatory frameworks, such as Basel II for banking and Solvency 2 for insurance, mandate a focus on operational risk. A fast growing literature exists on the various aspects of operational risk modelling; see the list of references towards the end of the paper.

In this paper we discuss some of the more recent Extreme Value Theory (EVT) methodology which may be useful towards the statistical analysis of certain types of operational loss data. The key attraction of EVT is that it offers a set of ready-made approaches to the most difficult problem of operational risk analysis, that is how can risks that are both extreme and rare be modelled appropriately? Applying classical EVT to operational loss data however raises some difficult issues. The obstacles are not really due to a technical justification of EVT, but more to the nature of the data. As already explained in Embrechts, Furrer and Kaufmann (2003) and Embrechts, Kaufmann and Samorodnitsky (2004), whereas EVT is the natural set of statistical techniques for estimating high quantiles of a loss distribution, this can be done with sufficient accuracy only when the data satisfy specific conditions; we further need sufficient data to calibrate the models. In Embrechts, Furrer and Kaufmann (2003) we give a simulation study indicating the sample size needed in order to estimate reliably certain high quantiles, and this under ideal (so called iid) data structure assumptions. From the above two papers we can definitely infer that though "EVT is a highly useful tool for high-quantile estimation, the present data availability and data structure of operational risk losses make a straightforward EVT application highly questionable". Nevertheless, for specific subclasses where quantitative data can be reliably gathered, EVT offers a useful tool. However, even in these cases, one has to go beyond standard EVT to come up with a correct modelling. To illustrate the latter issue, consider Figure 1 taken from Embrechts, Kaufmann and Samorodnitsky (2004); we refer to that paper for a more detailed discussion of the data. For our purposes, it suffices to recall that the data span a 10 year period for three different operational risk loss types, referred to as Types 1, 2 and 3. The stylised facts observed here are:

- the historical period is relatively short (only 10 years of data);
- loss amounts very clearly show extremes;

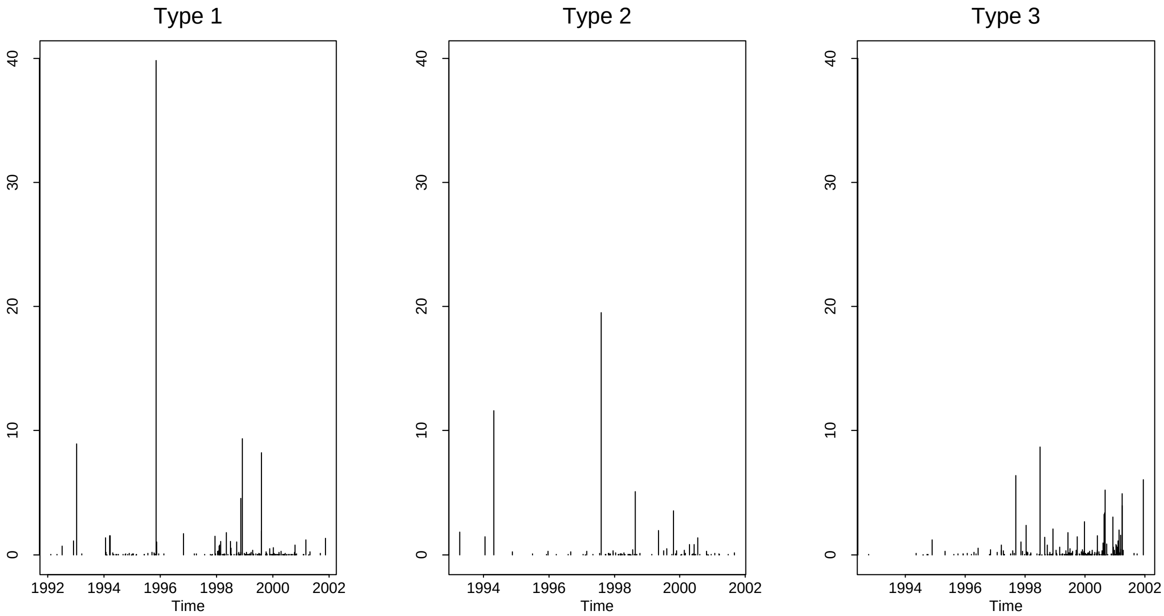

Figure 1: Operational risk losses. From left to right: Type 1 ($n = 162$), Type 2 ($n = 80$), Type 3 ($n = 175$).

- loss occurrence times are irregularly spaced in time, and
- the number of occurrences seems to increase over time with a radical change around 1998.

The last point very clearly highlights the presence of non-stationarity in operational loss data. The "discontinuity" might be due to the effort to build such a database of losses of the same type going back about 10 years; quantifying operational risk only became an issue in the later nineties. This is referred to as reporting bias. Such structural changes may also be due to an internal change (indogenous effect; management action, M&A) or changes in the economic/political/regulatory environment in which the company operates (exogenous effects).

In this paper, we adapt classical EVT to take both non-stationarity and covariate modelling (different types of losses) into account. Chavez-Demoulin (1999), Chavez-Demoulin and Davison (2004) contain the relevant methodology. Chavez-Demoulin and Embrechts (2004) explain the new technique for finance and insurance related applications. The paper is organised as follows. In Section 2, we briefly review the Peaks over Threshold (POT) method and the main operational risk measures to be analysed. In Section 3, the adapted classical POT method, taking non-stationarity and covariate modelling into account, is applied to the operational risk loss data from Figure 1.

2 The Basic EVT Methodology

Over the recent years, Extreme Value Theory has been recognized as a very useful set of probabilistic and statistical tools for the modelling of rare events and their impact in insurance, finance and quantitative risk management. Numerous publications have exemplified this point. Embrechts, Klüppelberg and Mikosch (1997) detail the mathematical theory with insurance and finance applications in mind. The edited volume Embrechts (2000) contains an early summary of EVT applications to risk management. Reiss and Thomas (2001) and Coles (2001) are very readable introductions to EVT in general.

Below, we only give a very brief introduction to EVT and in particular to the Peaks Over Threshold (POT) method for high-quantile estimation. A more detailed account is to be found in the references below; for our purpose, Chavez-Demoulin and Davison (2004) and Chavez-Demoulin and Embrechts (2004) contain methodological details.

From the latter paper, we borrow the basic notation (see also Figure 2):

- ground-up losses are denoted by $Z_1, Z_2, \ldots, Z_q$;
- u is a typically high threshold, and
- $W_1, \ldots, W_n$ are the excess losses from $Z_1, \ldots, Z_q$ above u, i.e. $W_j = Z_i - u$ for some $j = 1, \ldots, n$ and $i = 1, \ldots, q$, where $Z_i > u$.

Note that u is a pivotal parameter to be set by the modeller so that the excesses above u, $W_1, \ldots, W_n$, satisfy the required properties from the POT method; see Leadbetter (1991) for the basic theory and for instance Embrechts, Klüppelberg and Mikosch (1997) for an overview of the method. For iid losses, the excesses $W_1, \ldots, W_n$, asymptotically for n large, follow a so-called Generalized Pareto Distribution (GPD):

$$G_{\kappa,\sigma}(w) = \begin{cases} 1 - (1 + \kappa w/\sigma)_+^{-1/\kappa}\,, & \kappa \neq 0, \\ 1 - \exp(-w/\sigma)\,, & \kappa = 0\,. \end{cases}$$

For operational loss modelling one typically finds $\kappa > 0$ which corresponds to ground-up losses $Z_1, \ldots, Z_q$ following a Pareto-type distribution with power tail with index

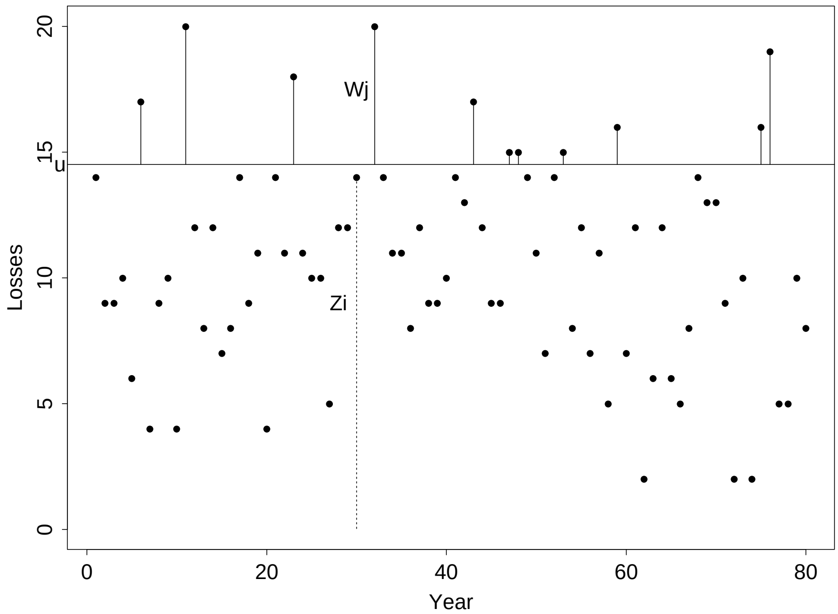

Figure 2: The point process of exceedances (POT).

$1/\kappa$, i.e.

$$P(Z_i > z) \sim z^{-1/\kappa} L(z), \qquad z \to \infty,$$

for some slowly varying function L; see Embrechts, Klüppelberg and Mikosch (1997). From Leadbetter (1991) it also follows that for u high enough, the exceedance points of $Z_1, \dots, Z_q$ of the threshold u follow (approximately) a homogeneous Poisson process with intensity $\lambda > 0$. Based on Leadbetter (1991), an approximate log-likelihood function $l(\lambda, \sigma, \kappa)$ can be derived; see Chavez-Demoulin and Embrechts (2004) for details. In a further step, the POT method can be extended by allowing the parameters λ, σ, κ to be dependent on time and explanatory variables so as to allow for non-stationarity; this is very important for the applications to operational risk modelling. In the next section (where we apply the POT method to the data in Figure 1), we will take for $\lambda = \lambda(t)$ a specific function of time which models the obvious "increase" in loss intensity in Figure 1. We moreover will differentiate between the different loss types and adjust the parameters κ and σ accordingly.

Before we proceed with the data analysis, we briefly review the main risk measures to be analysed, Value-at-Risk (VaR) and Expected-Shortfall (ES) (also referred to as "conditional VaR", "mean excess loss", "beyond VaR" or "tail VaR"). The ES is an alternative risk measure that has been proposed to alleviate some conceptual problems inherent in VaR. For α close to 1 and a general loss random variable X with distribution function F, these measures are defined as follows:

$$\mathrm{VaR}_\alpha = \mathrm{F}^{-1}(1-\alpha),$$

$$\mathrm{ES}_\alpha = \mathrm{E}(\mathrm{X} \mid \mathrm{X} > \mathrm{VaR}_\alpha).$$

In cases where the POT method can be applied, these measures can be estimated as follows:

$$\widehat{\mathrm{VaR}}_\alpha = u + \frac{\hat{\sigma}}{\hat{\kappa}} \left\{ \left(\frac{1-\alpha}{\hat{\lambda}} \right)^{-\hat{\kappa}} - 1 \right\}, \tag{1}$$

and

$$\widehat{\mathrm{ES}}_\alpha = \left\{ \frac{1}{1-\hat{\kappa}} + \frac{\hat{\sigma} - \hat{\kappa} u}{(1-\hat{\kappa})\widehat{\mathrm{VaR}}_\alpha} \right\} \widehat{\mathrm{VaR}}_\alpha. \tag{2}$$

Here $\hat{\lambda}, \hat{\kappa}, \hat{\sigma}$ are the maximum likelihood estimators of λ, κ and σ. Interval estimates can be obtained by the delta method or by the profile likelihood approach and has been programmed into the freeware EVIS by Alexander McNeil, available under www.math.ethz.ch/˜mcneil.

Though an analysis of the data in Figure 1 in Section 3 is self-contained, the interested reader, wanting to learn more about the specifics of modelling non-stationarity and covariates into the POT method is adviced to read Chavez-Demoulin and Embrechts (2004) and the references therein before proceeding. The less technical reader will no doubt find the analysis presented in the next section sufficiently easy to follow in order to grasp the relevance of this more advanced EVT method.

3 POT analysis of the operational loss data

In the previous sections, we briefly laid the foundation of the approach towards the analysis of extremes based on the exceedances of a high threshold. We now return to the operational risk data of Figure 1 which consists of three different types over a 10 year period. Our analysis below is more illustrative; in order to become fully applicable, much larger operational loss data bases will have to become available. From the discussion of the data, it follows that we should at least take the risk type T as well as the non-stationarity (switch around 1998) into account. Following Embrechts, Kaufmann and Samorodnitsky (2004), we pool the data in the three panels of Figure 1 to get a sample size bigger than when analysing each loss data separately. Using the advanced POT modelling, including non-stationarity and covariates, the data pooling has also the advantage to allow for testing interaction between explanatory variables (is there an interaction between type of loss and regime switching, say?). In line with Chavez-Demoulin and Embrechts (2004), we fix a threshold $u = 0.4$ and concentrate on the VaR and ES estimation. The latter paper also contains a sensitivity analysis of the results with respect to this choice of threshold u. A result from that analysis is that small variations in the value of the threshold have nearly no impact. So concretely, we want to model VaR_α and ES_α as functions of time: are they constant or changing in time? Are they dependent on the type of losses? And if the latter is the case, how do they change with time? Following the non-parametric methodology summarized in Chavez-Demoulin and Embrechts (2004), we fit different models for λ, κ and σ allowing for:

- functional dependence on *time* $g(t)$, where t refers to the *year* over the domain of study;
- dependence on T, where T defines the *Type* of loss data through an indicator I_T:

$$I_T = \begin{cases} 1, & \text{if} \quad Type = T, \\ 0, & \text{otherwise}, \end{cases}$$

  with $T = 1, 2, 3$, and
- *discontinuity* modelling through an indicator $I_{(t>t_c)}$ where $t_c = 1998$ is the year of change point or regime switching and

$$I_{(t>t_c)} = \begin{cases} 1, & \text{if} \quad t > t_c, \\ 0, & \text{if} \quad t \le t_c. \end{cases}$$

Of course a more formal test on the existence and value of t_c can be included; the rather pragmatic choice of $t_c = 1998$ suffices for this first illustrative analysis. We apply the different possible models to each parameters λ, κ and σ and compare them (using tests based on the likelihood ratio statistics).

The selected model for the Poisson intensity $\lambda(t, T)$ is

$$\log \hat{\lambda}(t, T) = \hat{\gamma}_T I_T + \hat{\beta} I_{(t>t_c)} + \hat{g}(t).$$

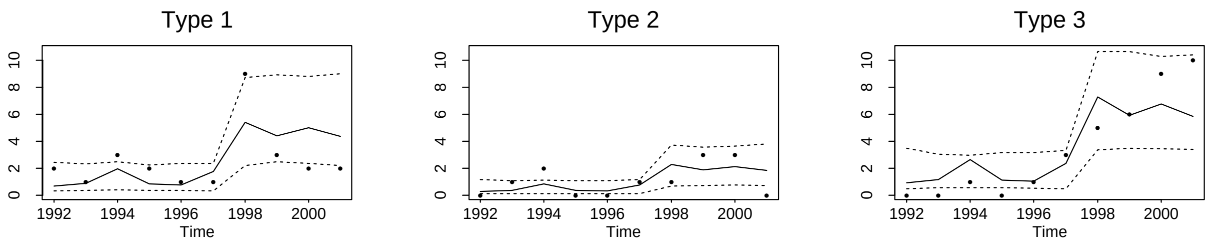

Figure 3: Operational risk losses. From left to right: Estimated Poisson intensity $\hat{\lambda}$ and 95% confidence intervals for data loss of Type 1, 2, 3. The points are the yearly numbers of exceedances over $u = 0.4$.

Inclusion of the first component $\hat{\gamma}_T I_T$ on the right hand side indicates that the type of loss T is important to model the Poisson intensity; that is the number of exceedances over the threshold differs significantly for each type of loss 1, 2 or 3. The selected model also contains the discontinuity indicator $I_{(t>t_c)}$ as a test based on the hypothesis that the model $\beta = 0$ suffices is rejected at a 5% level. We find $\hat{\beta} = 0.47(0.069)$ and the intensity is rather different in mean before and after 1998. Finally, it is clear that the loss intensity parameter λ is dependent on time (year). This dependence is modelled through the estimated function $\hat{g}(t)$. For the reader interested in fitting details, we use a smoothing spline with 8 degrees of freedom selected by AIC (see Chavez-Demoulin and Embrechts (2004)). Figure 3 represents the resulting estimated intensity $\hat{\lambda}$ for each type of losses and its 95% confidence interval based on bootstrap resampling schemes (details in Chavez-Demoulin and Davison (2004)). The resulting curves seem to capture the behaviour of the number of exceedances (points of the graphs) for each type rather well. The global increase of the estimated intensity curves therefore seems in accordance with reality. Note that the inclusion of the time dependent function $g(t)$ allows us to model this non-stationarity. The advantage of such a non-parametric technique becomes very clear.

Similarly, we fit several models for the GPD parameters $\kappa = \kappa(t, T)$ and $\sigma = \sigma(t, T)$ modelling the loss-size and compare them. For both κ and σ, the model selected depends only on the type of the losses, not on time t. Their estimates $\hat{\kappa}(T)$ and $\hat{\sigma}(T)$ and 95% confidence intervals are given in Figure 5. The shape parameter κ (upper panels) is around 0.7 for types 1 and 2 and is significantly smaller for type 3 (estimated value around 0.3); this suggests a loss distribution for type 3 with less heavy tail than for types 1 and 2. The effect due to the switch in 1998 is not retained in the models

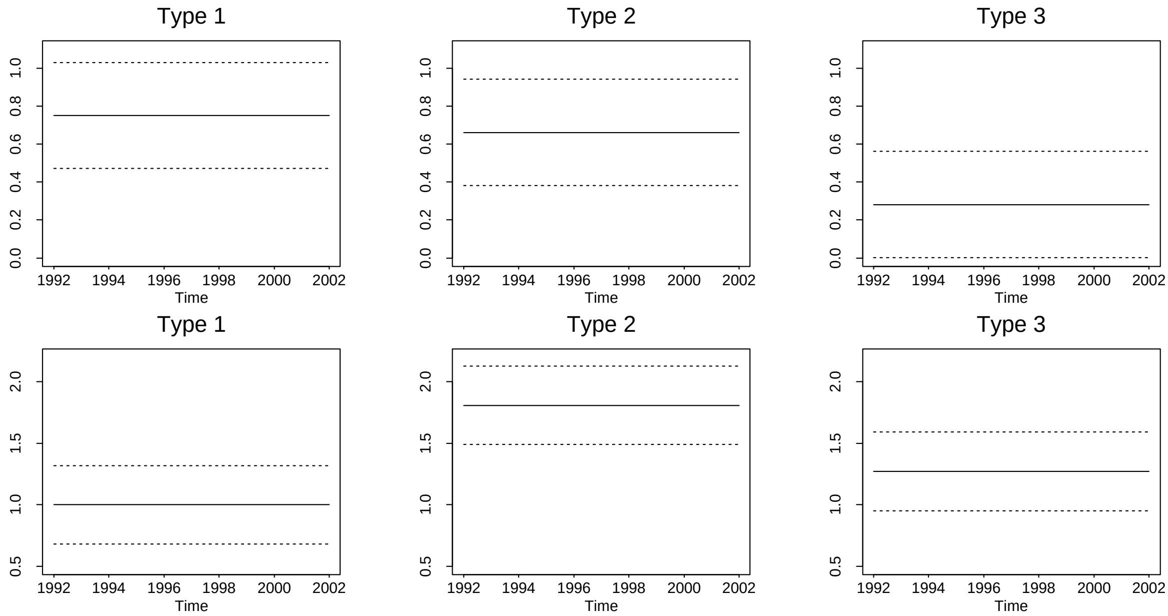

Figure 4: Operational risk losses. Estimated GPD parameters: upper $\hat{\kappa}$, lower $\hat{\sigma}$ and 95% confidence intervals for different loss types.

for κ and σ, i.e. the loss size distribution does not switch around 1998. Finally, note that, as the GPD parameters κ and σ are much more difficult to estimate than λ, the lack of sufficient data makes the detection of any trend and/or periodic components difficult.

To assess the model goodness-of-fit for the GPD parameters, a possible diagnostic can be based on the result that, when the model is correct, the residuals

$$R_j = \hat{\kappa}^{-1} \log \{1 + \hat{\kappa} W_j / \hat{\sigma}\}, \qquad j = 1, \ldots, n,$$

are approximately independent, unit exponential variables. Figure 4 shows exponential quantile plots for the residuals using the estimates $\hat{\kappa}(T)$ and $\hat{\sigma}(T)$ for the three types of loss data confounded. This plot suggests that our model is reasonable.

We now want to estimate the 99% VaR and the 99% ES at time 2002. Again, this estimation is illustrative as in practice (Basel II) values of the order of 99.97% are used for the calculation of operational risk measures. The $\mathrm{ES}_{0.99}$ at time 2002 is the conditional expectation of total loss over 2002 given that the loss is beyond the $\mathrm{VaR}_{0.99}$ level. Using our modelling approach for $\lambda(t, T)$, $\kappa(t, T)$ and $\sigma(t, T)$, one can predict the values of $\lambda(t + 1, T)$, $\kappa(t + 1, T)$ and $\sigma(t + 1, T)$ for each type T. Using equations (1) and (2) where $u = 0.4$ and λ, κ, σ are replaced by their predicted values

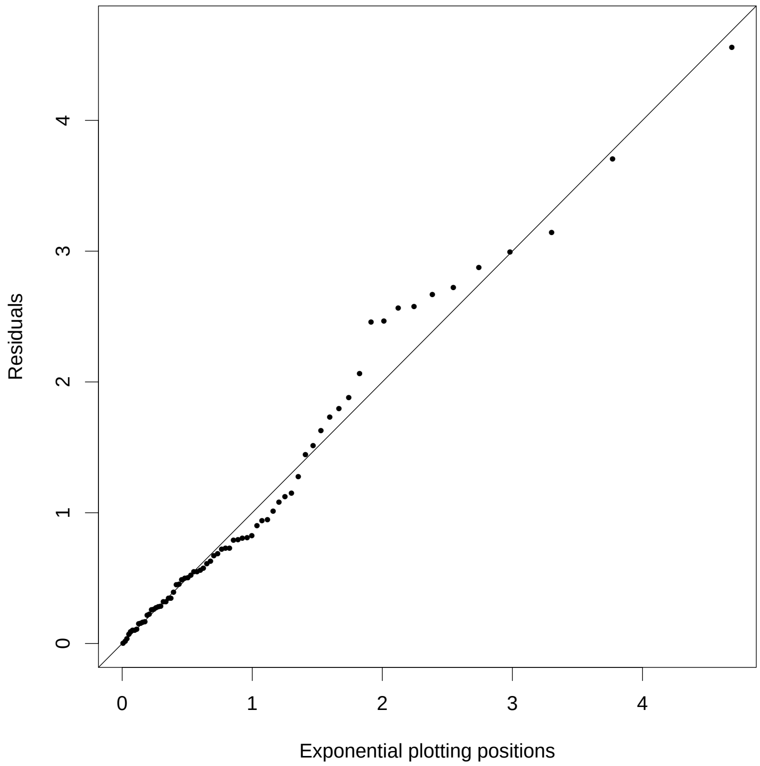

Figure 5: Operational risk losses. Residuals against exponential plotting positions.

$\hat{\lambda}(t+1,T)$, $\hat{\kappa}(T)$ and $\hat{\sigma}(T)$ (for $t = 2001$ and $T = 1,2,3$), it is then possible to estimate the 99% VaR and 99% ES at time 2002. As the model selected for λ depends on time t, the risk measures are "dynamic" (prefix d below), and as the selected models also depend on the type of losses (index T below), we denote $\mathrm{dVaR}^T_{0.99}(t)$ and $\mathrm{dES}^T_{0.99}(t)$ the estimated risk measures at time t for type T. Table 1 provides the 99% VaR and 99% ES estimated for each type of losses 1,2,3 at time 2002 ($\widehat{\mathrm{dVaR}}^T_{0.99}(2002)$ and $\widehat{\mathrm{dES}}^T_{0.99}(2002)$). The values in brackets are the 95% bootstrap confidence intervals bounds (the missing values in the confidence intervals are due to a lack of data to estimate very heavy tails as for types 1 and 2). For instance, $\widehat{\mathrm{dVaR}}^{T=1}_{0.99}(2002)$ gives an estimation of the total 2002 loss of type 1 of 40.4 at 99% confidence. We also note that this value for type 3 is around 12, significantly smaller than the estimated losses for types 1 and 2. The importance of using models including covariates (representing type) instead of pooling the data and finding a unique estimate value of VaR (or ES) is highlighted here. In a certain sense, the use of our adapted model allows to exploit all the provided information about the data, a feature which is becoming more and more

	$\widehat{\mathrm{dVaR}}^{T}_{0.99}(2002)$	$\widehat{\mathrm{dES}}^{T}_{0.99}(2002)$
$T = 1$	40.4 $(17.3, 120.5)$	166.4 $(-, -)$
$T = 2$	48.4 $(11.9, 83.7)$	148.5 $(21.4, -)$
$T = 3$	11.9 $(7.2, 27.5)$	18.8 $(9.8, 63.8)$

Table 1: Operational risk losses. Estimated 99% dynamic VaR and ES for each type of losses over 2002. The values in brackets are the 95% confidence intervals bounds.

crucial, particularly in the context of operational and credit risk. Using the estimated historical VaR values, it is possible to test whether the hypothesis that the approach correctly estimates the risk measures holds. This backtesting test however would need, in our case, much more historical.

4 Comment

With the increasing interest on explicit treatment of operational risk (Basel II and Solvency 2), there is a pressing need for flexible modelling of severe tail loss events. The use of an adapted extreme value method taking into account non-stationarity (time dependent structure) and covariates (changing business and/or economic environment) provides a convenient, rapid and flexible explorative technique that will have the ability to self-improve with the further growth of data-bases. It also puts into evidence features of the underlying distribution as the covariates changes, provides an objective tool to determine their relative importance and highlights (unexpected) interactions of risk components. We stress once more that much longer databases are needed to make our approach fully operational.

Acknowlegements

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