Liquidity and Risk Management

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Risk management plays a central role in institutional investors' allocation of capital to trading. For instance, Jorion (2000, page xxiii) states that value-at-risk (VaR) "is now increasingly used to allocate capital across traders, business units, products, and even to the whole institution." This paper studies how risk management practices can affect market liquidity and prices. We first show that tighter risk management leads to lower market liquidity, in that it takes longer to find a buyer with unused risk-bearing capacity, and, since liquidity is priced, to lower prices.

Furthermore, not only does risk management affect liquidity, but liquidity can also affect risk-management practices. For instance, BIS (2001, page 15) states that "For the internal risk management, a number of institutions are exploring the use of liquidity adjusted-VaR, in which the holding periods in the risk assessment are adjusted to account for market liquidity, in particular by the length of time required to unwind positions."

We show that if traders are subject to such a liquidity-adjusted VaR, then a multiplier effect arises: tighter risk management leads to longer expected selling times, implying higher

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risk over the expected selling period, which further tightens the risk management, and so on. This snowballing feedback between liquidity and risk management can help explain liquidity crisis and may also lead to spillover across markets, although we do not model this.

We capture these effects by extending the search model for financial securities of Duffie, Gârleanu, and Pedersen (2005, 2006) (DGP). While DGP relied on exogenous position limits, we endogenize positions based on a risk-management constraint, which can be either a simple or a liquidity-adjusted VaR. Weill (2003) considers another extension of DGP in which marketmakers' liquidity provision is limited by capital constraints. Our multiplier effect is similar to that of Brunnermeier and Pedersen (2006) who show that liquidity and traders' margin requirements can be mutually reinforcing.

1 Model

The economy has two securities: a "liquid" security with risk-free return r (i.e. a "money-market account"), and a risky illiquid security. The risky security has a dividend-rate process X and a price $P(X)$, which is determined in equilibrium. The dividend rate is Lévy with finite second moment. It has a constant drift normalized to zero, $E_t\left(X(t+T)-X(t)\right)=0$, and a volatility $\sigma_X>0$,

$$\operatorname{var}_t\left(X(t+T)-X(t)\right) \quad = \quad \sigma_X^2\,T. \tag{1}$$

Examples include Brownian motions, simple and compound Poisson processes, and sums of these.

The economy is populated by a continuum of agents who are risk neutral and infinitely lived, have a time-preference rate equal to the risk-free interest rate $r > 0$, and must keep their wealth bounded from below. Each agent is characterized by an intrinsic type $i \in \{h, l\}$, which is a Markov chain, independent across agents, and switching from l ("low") to h ("high") with intensity λ_u, and back with intensity λ_d. An agent of type i holding θ_t shares of the asset incurs a holding cost of $\delta > 0$ per share and per unit of time if he violates his *risk-management constraint*

$$\mathrm{var}_t\left(\theta_t\left[P(X_{t+\tau}) - P(X_t)\right]\right) \leq (\sigma^i)^2, \tag{2}$$

where σ^i is the risk-bearing capacity, defined by $\sigma^h = \bar{\sigma} > 0$ and $\sigma^l = 0$.

We use this constraint as a parsimonious way of capturing VaR constraints, which are used by most financial institutions.[1] The low risk-bearing capacity of the low-type agents can be interpreted as a need for more stable earnings, hedging reasons to sell the asset, high financing costs, or a need for cash (e.g., an asset manager whose investors redeem capital).

We consider two types of risk management: (a) "simple risk management," in which the variance of the position in (2) is computed over a fixed time horizon τ, and (b) "liquidity-adjusted risk management," in which the risk is computed over the time required for selling the asset, which will be a random equilibrium quantity.

Because agents are risk neutral and we are interested in a steady-state equilibrium, we restrict attention to equilibria in which, at any given time and state of the world, an agent

[1]A VaR constraint stipulates that $Pr\left(\theta\left[P(X_{t+T}) - P(X_t)\right] \geq VaR\right) \leq \pi$ for some risk limit VaR and some confidence level π. If X is a Brownian motion, this is the same as (2). We note that rather than considering only price risk, we could alternatively consider the risk of the gains process (i.e. including dividend risk) $G_{t,u} = P(X(t+u)) - P(X(t)) + \int_t^{t+u} X(s)\,ds$. This yields qualitatively similar results (and quantitatively similar for many reasonable parameters since dividend risk is orders of magnitude smaller than price risk over a small time period).

holds either 0 or $\bar{\theta}$ units of the asset, where $\bar{\theta}$ is the largest position that satisfies (2) with $\sigma^i = \bar{\sigma}$, taking the prices and search times as given.[2] Hence, the set of agent types is $\mathcal{T} = \{ho, hn, lo, ln\}$, with the letters "$h$" and "$l$" designating the agent's current intrinsic risk-bearing state as high or low, respectively, and with "o" or "n" indicating whether the agent currently owns $\bar{\theta}$ shares or none, respectively. We let $\mu_\zeta(t)$ denote the fraction at time t of agents of type $\zeta \in \mathcal{T}$. Clearly, these fractions add up to 1:

$$1 = \mu_{ho} + \mu_{hn} + \mu_{lo} + \mu_{ln}. \tag{3}$$

Further, equating the exogenous total supply of shares per investor, denoted by Θ, with the holding of owners gives

$$\Theta = \bar{\theta}(\mu_{ho} + \mu_{lo}). \tag{4}$$

Central to our analysis is the notion that the risky security is not perfectly liquid, in the sense that an agent can only trade it when she finds a counterparty. Every agent finds a potential counterparty, selected randomly from the set of all agents, with intensity λ, where $\lambda \geq 0$ is an exogenous parameter characterizing the market liquidity for the asset. Hence, the intensity of finding a type-ζ investor is $\lambda\mu_\zeta$, that is, the search intensity multiplied by the fraction of investors of that type. When two agents meet, they bargain over the price, with the seller having bargaining power $q \in [0, 1]$.

This model of illiquidity captures directly the search that characterizes OTC markets. It might also capture delays in reaching trading decisions and in mobilizing capital in specialist

[2] Note that the existence of such an equilibrium requires that the risk limit $\bar{\sigma}$ be not too small relative to the total supply Θ, a condition that we assume throughout.

and electronic limit-order-book markets, although these markets are, of course, distinct from OTC markets.

2 Equilibrium Risk Management, Liquidity, and Prices

We now proceed to derive the steady-state equilibrium agent fractions μ, the maximum holding $\bar{\theta}$, and the price P. Naturally, low-type owners lo want to sell and high-type non-owners hn want to buy. These types of agents trade when they meet and, together with the type switches, this leads to the following steady state equations for the masses.

$$\begin{aligned}
0 &= \dot{\mu}_{lo}(t) &&= -2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_u\mu_{lo}(t) + \lambda_d\mu_{ho}(t) \\
0 &= \dot{\mu}_{hn}(t) &&= -2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_d\mu_{hn}(t) + \lambda_u\mu_{ln}(t) \\
0 &= \dot{\mu}_{ho}(t) &&= 2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_d\mu_{ho}(t) + \lambda_u\mu_{lo}(t) \\
0 &= \dot{\mu}_{ln}(t) &&= 2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_u\mu_{ln}(t) + \lambda_d\mu_{hn}(t).
\end{aligned} \tag{5}$$

The first of these equations states that the change in the fraction of lo agents depends on three things, corresponding to the three terms on the right hand side: First, whenever an lo agent meets an hn investor, he sells his asset and is no longer an lo agent. Second, whenever the intrinsic type of a lo agent switches to high, he becomes an ho agent. Third, ho agents can switch type and become lo. Duffie, Gârleanu, and Pedersen (2005) show that there is a unique stable steady-state solution to (3)–(5), taking $\bar{\theta}$ as fixed. Here, agents' positions $\bar{\theta}$ are endogenous — indeed, they depend on risk management, which depends on prices, which depend on agent fractions — so that, ultimately, one must find a fixed point.

Agents take the steady-state distribution μ as fixed when they derive their optimal strategies and utilities for remaining lifetime consumption, as well as the bargained price P. The utility of an agent depends on his current type, $\zeta(t) \in \mathcal{T}$, (i.e. whether he is a high or a low type and whether he owns zero or $\bar{\theta}$ shares), the current dividend $X(t)$, and the wealth $W(t)$ in his bank account:

$$V_\zeta(X(t), W_t) = W_t + 1_{(\zeta \in \{ho, lo\})} \frac{\bar{\theta} X(t)}{r} + \bar{\theta} v_\zeta, \tag{6}$$

where the type-dependent utility coefficients v_ζ are to be determined. The price is determined through bilateral bargaining. A high-type non-owner pays at most his reservation value $\Delta V_h = V_{ho} - V_{hn}$ for obtaining the asset, while a low-type owner requires a price of at least $\Delta V_l = V_{lo} - V_{ln}$. Nash bargaining results in the price

$$P \;=\; \Delta V_l (1-q) + \Delta V_h \, q\,, \tag{7}$$

where q is the bargaining power of the seller. We conjecture, and later confirm, that the equilibrium asset price per share is of the form

$$P(X(t)) = \frac{X(t)}{r} + p, \tag{8}$$

for a constant p to be determined. The value-function coefficients and p are given by the Hamilton-Jacobi-Bellman equations solved in the appendix to give

Proposition 1 *If the risk-limit $\bar{\sigma}$ is sufficiently large, there exists an equilibrium with holdings 0 and $\bar{\theta}$ that satisfy the risk management constraint (2) with equality for low- and* *high-type agents, respectively. With simple risk management, the equilibrium is unique and*

$$\bar{\theta} = \frac{r\bar{\sigma}}{\sigma_X} \frac{1}{\sqrt{\tau}}. \tag{9}$$

With liquidity-adjusted risk management, $\bar{\theta}$ depends on the equilibrium fraction of potential buyers μ_{hn} and satisfies

$$\bar{\theta} = \frac{r\bar{\sigma}}{\sigma_X} \sqrt{\lambda \mu_{hn}}. \tag{10}$$

In both cases, the equilibrium price is given by

$$P(X_t) = \frac{X_t}{r} - \frac{\delta}{r} \frac{r(1-q) + \lambda_d + 2\lambda\mu_{lo}(1-q)}{r + \lambda_d + 2\lambda\mu_{lo}(1-q) + \lambda_u + 2\lambda\mu_{hn} q}, \tag{11}$$

where the fractions of agents μ depend on the type of risk management.

These results are intuitive. The "position limit" $\bar{\theta}$ increases in the risk limit $\bar{\sigma}$ and decreases in the asset volatility and in the square root of the VaR period length, which is τ under simple risk management and $1/\lambda\mu_{hn}$ under liquidity-adjusted risk management. In the latter case, position limits increase in the search intensity λ and in the fraction of eligible buyers μ_{hn}.

The price is the present value of dividends, $\frac{X_t}{r}$, minus a discount for illiquidity. Naturally, the liquidity discount is larger if there are more low-type owners in equilibrium (μ_{lo} is larger) and fewer high-type non-owners ready to buy (μ_{hn} is smaller).

Of the equilibria obtaining with liquidity-adjusted risk management, we concentrate on the ones that are stable, in the sense that increasing $\bar{\theta}$ marginally would result in equilibrium quantities violating the VaR constraint (2). Conversely, an equilibrium is unstable if a marginal change in holdings that violates the constraint would result in the equilibrium adjusting so that the constraint is not violated. If an equilibrium exists, then a stable equilibrium exists – in particular, the equilibrium with the largest $\bar{\theta}$ among all equilibria, which also has the highest welfare of all equilibria, is stable.

The main result of the paper characterizes the equilibrium impact of the underlying dividend volatility:

Proposition 2 *Suppose that $\bar{\sigma}$ is large enough for existence of an equilibrium. Consider a stable equilibrium with liquidity-adjusted risk management and let $\tau = \frac{1}{\lambda \mu_{hn}}$, which means that the equilibrium allocations and price are the same with simple risk management. The following holds.*

With higher dividend volatility σ_X, keeping all other parameters the same, (i) the equilibrium positions $\bar{\theta}$ decreases, (ii) expected search times for selling increase, and (iii) prices decrease. All three effects are larger with liquidity-adjusted risk management.

The negative dependence of the price on the volatility of the asset is a liquidity effect, brought about by a reduction in the risk-bearing capacity of an investor relative to the total risk to be held. A larger volatility thus implies a smaller quantity of agents whose risk capacities qualify them to buy the asset (that is, fewer liquid investors who do not already own the asset), which increases the search time for sellers. This worsens their bargaining position and prices go down.

Importantly, with liquidity-adjusted risk management, increased search times for sellers further tighten the risk management constraint, thus further reducing positions, further

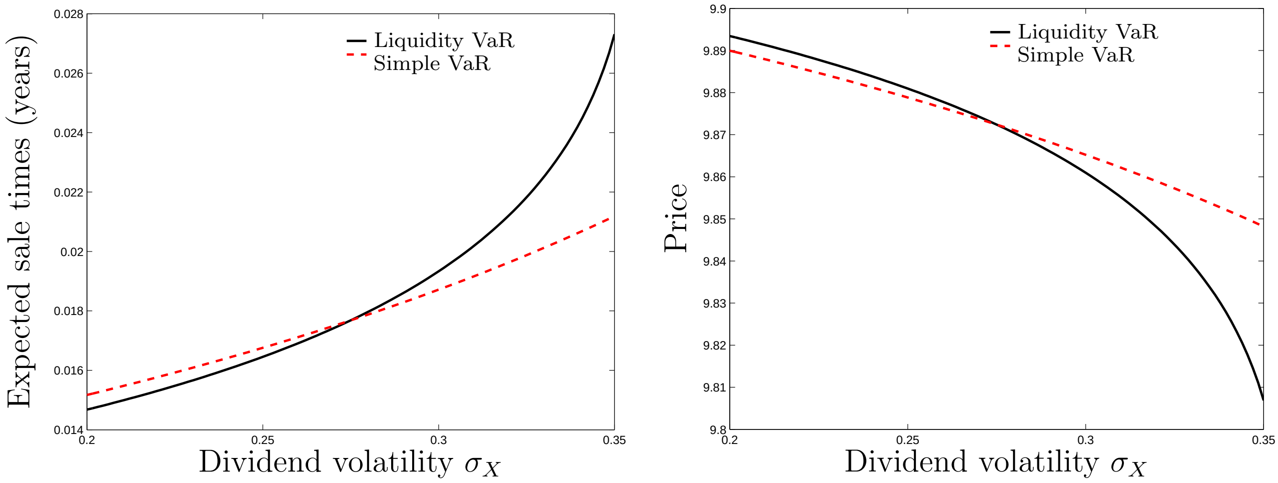

Figure 1: The effects of dividend volatility on equilibrium seller search times (left panel) and prices (right panel) with, respectively simple (dashed line) and liquidity-adjusted (solid line) risk management.

increasing search times, and so on indefinitely. This multiplier arising from the feedback between trading liquidity and risk management clearly magnifies the effect of increased volatility. Similarly, this multiplier also increases the sensitivity of the economy with liquidity-adjusted risk management to other shocks.

We illustrate our model with a numerical example in which $\lambda = 100$, $r = 10\%$, $X_0 = 1$, $\lambda_d = 0.2$, $\lambda_u = 2$, $\delta = 3$, $q = 0.5$, $\Theta = 0.8$, and $\bar{\sigma} = 85\%$. Figure 1 shows how prices (right panel) and sellers' expected search times (left panel) depend on asset volatility. The solid line shows this for liquidity-adjusted risk management and the dashed for simple risk management with $\tau = 0.0177$, which is chosen so that the risk management schemes are identical for $\sigma_X = 27.5\%$. Search times increase and prices decrease with volatility. Importantly, these sensitivities are stronger (i.e., the curves are steeper) with liquidity-adjusted risk management due to the interaction between market liquidity (i.e., search times) and risk management.

A Proofs

Proof of Proposition 1:

The value function coefficients are given by

$$
\begin{aligned}
0 &= rv_{lo} - \lambda_u(v_{ho} - v_{lo}) - 2\lambda\mu_{hn}(p - v_{lo} + v_{ln}) + \delta \\
0 &= rv_{ln} - \lambda_u(v_{hn} - v_{ln}) \\
0 &= rv_{ho} - \lambda_d(v_{lo} - v_{ho}) \\
0 &= rv_{hn} - \lambda_d(v_{ln} - v_{hn}) - 2\lambda\mu_{ho}(v_{ho} - v_{hn} - p) \\
p &= (v_{lo} - v_{ln})(1 - q) + (v_{ho} - v_{hn})q.
\end{aligned}
\tag{12}
$$

The first equation means that an agent of type lo has a zero change in steady-state utility. The change in his utility is due to opportunity cost $-rv_{lo}$, expected change in intrinsic-type $\lambda_u(v_{ho}-v_{lo})$, trade $2\lambda\mu_{hn}(p-v_{lo}+v_{ln})$, and holding cost $-\delta$. The other equations are similar. Direct solution of this system yields

$$
p = \frac{\delta}{r}\frac{r(1-q) + \lambda_d + 2\lambda\mu_{lo}(1-q)}{r + \lambda_d + 2\lambda\mu_{lo}(1-q) + \lambda_u + 2\lambda\mu_{hn}q}\,. \tag{13}
$$

Given the dependence of $P(X_t)$ on X_t, it is immediate that

$$
\mathrm{var}_t(P(X_{t+\tau}) - P(X_t)) = \frac{\sigma_X^2}{r^2}\tau
$$

for constant τ. If τ is randomly distributed with constant arrival intensity $\lambda\mu_{hn}$,

$$\begin{aligned}
\mathrm{var}_t(P(X_{t+\tau}) - P(X_t)) &= \frac{1}{r^2}\mathrm{var}_t(X_{t+\tau} - X_t) \\
&= \frac{1}{r^2}\left[E_t(\mathrm{var}_t(X_{t+\tau} - X_t|\tau)) + \mathrm{var}_t(E_t(X_{t+\tau} - X_t|\tau))\right] \\
&= \frac{1}{r^2}[\sigma_X^2 E_t(\tau)] \;=\; \frac{\sigma_X^2}{r^2\lambda\mu_{hn}},
\end{aligned}$$

and it is clear that, when the VaR constraint (2) binds, the equilibrium holding θ is given by (9) or (10), depending on the nature of risk management.

$\square$

Proof of Proposition 2: The equilibrium with the two kinds of risk management is given by $f_i(\bar\theta) = \frac{r\bar\sigma}{\sigma_X}$, where $f_0(\bar\theta) = \bar\theta\sqrt{\tau}$ and $f_1(\bar\theta) = \frac{\bar\theta}{\sqrt{\lambda\mu_{hn}(\bar\theta)}}$ Clearly, $f_0 = f_1$ when $\tau = \frac{1}{\lambda\mu_{hn}}$, so that the two equilibria are identical.

The sensitivity $\bar\theta'$ of the equilibrium position $\bar\theta$ to the asset volatility σ_X is given by

$$f_i'\bar\theta' \;=\; -\frac{r\bar\sigma}{\sigma_X^2}. \tag{14}$$

With simple risk management, it is clear that $f_0' > 0$, so that the equilibrium position decreases in the volatility σ_X. A decreasing $\bar\theta$ further leads to an increasing expected search time for sellers $1/(\lambda\mu_{hn})$ and a decreasing price, because $\partial\mu_{hn}/\partial\bar\theta > 0$ and $\partial\mu_{lo}/\partial\bar\theta < 0$. The latter results follow from the equilibrium condition

$$0 \;=\; 2\lambda\mu_{hn}^2 + \left(2\lambda\left(\frac{\Theta}{\bar\theta} - \frac{\lambda_u}{\lambda_d+\lambda_u}\right) + \lambda_u + \lambda_d\right)\mu_{hn} - \lambda_u\left(1 - \frac{\Theta}{\bar\theta}\right).$$

With liquidity-adjusted risk management, $f_1' > 0$ by the definition of a stable equilibrium, and, since $\partial\mu_{hn}/\partial\bar{\theta} > 0$, $f_1' < f_0'$. Hence, with liquidity-adjusted risk management, the effects of σ_X on the equilibrium quantities are larger in absolute value and of the same sign as in the case of simple risk management. A stable equilibrium exists because $f_1 < \infty$ on (Θ, ∞), while $\lim_{x\to\Theta} f_1(x) = \lim_{x\to\infty} f_1(x) = \infty$, given that $\mu_{hn}(\Theta) = 0$ and $\lim_{x\to\infty} \mu_{hn}(x) > 0$.

$\square$

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