Importance Sampling for Integrated Market and Credit Portfolio Models

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Abstract:
Standard credit portfolio models do not model market risk factors, such as risk-free interest rates or credit spreads, as stochastic variables. Various studies have documented that a severe underestimation of economic capital can be the consequence. However, integrating market risk factors into credit portfolio models increases the computational burden of computing credit portfolio risk measures. In this paper, the application of various importance sampling techniques to an integrated market and credit portfolio model are presented and the effectiveness of these approaches is tested by numerical experiments. The main result is that importance sampling *can* reduce the standard error of the percentile estimators, but it is rather difficult to make statements about when the IS approach is especially effective. Besides, the combination of importance sampling techniques originally developed for pure market risk portfolio models with techniques originally developed for pure default mode credit risk portfolio models is less effective than simpler two step-IS approaches.

Keywords:
credit risk, importance sampling, interest rate risk, Value-at-Risk, variance reduction

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1 Introduction

Standard credit portfolio models do not model market risk factors, such as risk-free interest rates or credit spreads, as stochastic variables. Even the Basel II proposals do not intend to regulate the interest rate risk of the banking book in a quantitative way, but only qualitatively under pillar II.[1] For example in CreditMetrics, fixed income instruments, such as bonds or loans, are revalued at the risk horizon using currently observable forward rates for discounting future cash flows. Hence, the stochastic nature of the instrument's value which results from changes in factors other than credit quality is neglected. An additional consequence is that correlations between changes of the credit quality of the debtors and changes of market risk factors and, hence, the exposure at default cannot be integrated into the credit portfolio model. This is especially a problem for market-driven instruments, such as interest rate derivatives. Finally, correlations between the exposures at default of different instruments, which depend on the same or correlated market risk factors, cannot be modeled, either.

Various studies have documented that a severe underestimation of economic capital can be the consequence of the missing stochastic modeling of market risk factors, especially for high grade credit portfolios with a low stochastic dependence between the obligors' credit quality changes.[2] However, integrating market risk factors into credit portfolio models increases the computational burden of calculating portfolio risk measures. Most standard credit portfolio models rely on Monte Carlo simulations for computing the probability distribution of the future credit portfolio value, which can be quite time consuming already in the standard case, in particular when percentiles corresponding to high confidence levels are needed or when there are many obligors in the credit portfolio.

Hence, when adding market risk factors to standard credit portfolio models, the need for efficient methods for calculating credit risk measures is even more pressing than before. For

[1] See *Basel Committee on Banking Supervision* (2004).

[2] Studies, which analyze the effect of integrating an additional risk factor, such as stochastic interest rates or stochastic credit spreads, into a credit portfolio model are from *Kijima* and *Muromachi* (2000), *Barnhill* and *Maxwell* (2002), *Kiesel*, *Perraudin* and *Taylor* (2003) and *Grundke* (2004, 2005b). There are also first attempts to create an integrated market and credit risk portfolio framework for commercial credit portfolio models, for example that one developed by the risk management firm Algorithmics (see *Iscoe*, *Kreinin* and *Rosen* (1999)).

standard credit portfolio models, various efficiency enhancing computational approaches have been developed meanwhile. These can be broadly classified in four categories: First, approaches based on Monte Carlo simulations combined with variance reduction techniques have been presented. Most of these approaches employ importance sampling in order accelerate the computation of credit risk measures. Examples are *Xhiao* (2001), *Glasserman* and *Li* (2003a, b), *Kalkbrenner*, *Lotter* and *Overbeck* (2004) and *Merino* and *Nyfeler* (2004). An exception is the paper of *Tchistiakov*, *de Smet* and *Hoogbruin* (2004) who work with control variables. Second, Fourier based approaches have been described, e.g. by *Duffie* and *Pan* (2001), *Merino* and *Nyfeler* (2002), *Reiß* (2003) or *Grundke* (2005a). With the exception of the work of Duffie and Pan all these approaches suffer from the fact that the unconditional characteristic function of the credit portfolio value needed for the Fourier inversion formula can not be calculated in closed-form, but has still to be simulated. This is a potential drawback of the method because, depending on the portfolio composition, this can be quite time-consuming. Duffie and Pan can derive the unconditional characteristic function of the credit portfolio value in closed-form, but for this they need, among others, the assumption that a delta-gamma approximation of the credit portfolio value is sufficiently accurate which seems problematic for longer time horizons such as one year usually employed in the context of credit risk management. Third and fourth, computational approaches based on saddlepoint approximations (see e.g. *Martin*, *Thompson* and *Browne* (2001) or *Barco* (2004)) and granularity adjustments (see e.g. *Wilde* (2001), *Martin* and *Wilde* (2002), *Gordy* (2003) or *Pykhtin* (2004)) have been presented.

This paper fits best into the first category because it makes use of importance sampling as a special variance reduction technique when simulating the credit portfolio value at the risk horizon. The main contributions of this paper to the literature are twofold: First and most important, two drawbacks of previous papers (see above) about applications of variance reduction techniques to Monte Carlo simulations are overcome: All these approaches suffer from the fact that they do not consider market risk factors as relevant risk factors during the revaluation of the defaultable instruments at the risk horizon. Furthermore, the application of the methods in these papers is restricted to pure default mode credit portfolio models. Second, in this paper it is analyzed whether importance sampling techniques originally developed for pure market risk portfolio models can be combined with techniques originally developed for pure (default mode) credit risk portfolio models in order to decrease the variance of risk measure estimators. The effectiveness of the importance sampling techniques applied to integrated market and credit portfolio models is tested by means of numerical experiments.

The paper is structured as follows: In section 2 a framework for an integrated market and credit portfolio model is presented. Besides, a concrete specification of this general model is described which afterwards is used for the numerical experiments. Section 3 consists of the derivation of two importance sampling techniques for the general integrated market and credit portfolio model. The effectiveness of these approaches is tested by means of numerical experiments in section 4. Finally, in section 5 the main results are summarized and possible extensions of this study are outlined.

2 The Integrated Market and Credit Portfolio Model

2.1 General Approach

It is assumed that the credit portfolio consists of N market and credit risk sensitive instruments issued by N different corporates. The risk horizon, usually one year, is denoted by H and P is the real world probability measure. The possible credit qualities at the risk horizon are $1,\ldots,K$ where 1 denotes the best rating and K is the default state.

The central part of most standard credit portfolio models is the definition of the obligors' conditional default and transition probabilities. Denoting by $\eta_H^n \in \{1,\ldots,K\}$ the credit quality of obligor n at the risk horizon H and by η_0^n the respective rating at $t=0$, the conditional default (transition) probabilities are formally defined as:

$$P\left(\eta_H^n = k \middle| \eta_0^n = i, Z_1 = z_1,\ldots,Z_C = z_C\right) := f_{n,i,k}(z_1,\ldots,z_C) \tag{1}$$

with $f_{n,i,k}:\mathbb{R}^C \to [0,1]$, $k\in\{1,\ldots,K\}$, $i\in\{1,\ldots,K-1\}$, and $n\in\{1,\ldots,N\}$.

The set of variables $Z=(Z_1,\ldots,Z_C)^T \sim F^C$ are systematic credit risk factors that might be thought of as changes in equity indices or macro-economic variables within the risk horizon. These risk factors influence the credit quality changes of all obligors within the risk horizon. This vector is assumed to evolve according to the multivariate distribution F^C. Given the realization $(Z_1 = z_1,\ldots,Z_C = z_C)$ of the systematic credit risk factors and hence of the conditional default (transition) probabilities, credit quality changes of all obligors are assumed to be stochastically independent. This is the classical 'conditional independence'-framework for describing joint credit quality changes within a credit portfolio. Sampling from the N discrete distributions (1), the credit quality of all obligors at the risk horizon can be simulated for a specific scenario $(Z_1 = z_1, \dots, Z_C = z_C)$.

The price of the instrument i_n (e.g. defaultable (zero) coupon bonds or options with counterparty risk) at the risk horizon H, whose issuer n has not already defaulted before H and exhibits the rating $\eta_H^n \in \{1, \dots, K-1\}$, is denoted by

$$p_n(\eta_H^n; X_1, \dots, X_M; \mathrm{P_n}), \tag{2}$$

where the stochastic vector $\mathrm{X} = (X_1, \dots, X_M)^T \sim F^M$ represents the value of relevant market risk factors, such as e.g. risk-free interest rates, at the risk horizon. This vector is assumed to evolve according to the multivariate distribution F^M. $\mathrm{P_n}$ denotes a vector of additional parameters relevant for the pricing of the respective instrument i_n at the risk horizon. Note that the set of systematic credit risk factors $Z_1, \dots, Z_C$ and the set of market risk factors $X_1, \dots, X_M$ can overlap, e.g. if a risk-free interest rate is also a relevant credit risk driver. The joint distribution of the stochastic vector $(Z_1, \dots, Z_C; X_1, \dots, X_M)^T$ is denoted by F.

If the issuer n of the instrument i_n has already defaulted ($\eta_H^n = K$) before the risk horizon H, its value, in the case this value is positive, is set equal to a fraction δ of the value the instrument would have at the risk horizon when its issuer would be free of default risk. If the market value of this instrument is negative, nothing is changed because the bank whose credit portfolio is considered is a debtor of the defaulted issuer. The shape of the distribution of the recovery rate δ can vary with the seniority of a claim and the value of individual collaterals. For all defaulted issuers the recovery rate is drawn individually which ensures independence of the recovery rates across the different exposures. Usually, it is assumed that the recovery rate is beta-distributed and independent from all other stochastic variables of the respective model, such as the systematic credit risk drivers or the market risk factors, but it could also be a function of these risk factors.[3]

Finally, the value $\Pi(H)$ of the entire portfolio at the risk horizon H is just the sum over the individual values:

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3 See for example *Frye* (2000, 2003) or *Pykhtin* (2003).

$$\Pi(H) = \sum_{n=1}^{N} p_n(\eta_H^n; X_1,\ldots,X_M; \mathrm{P_n}) . \tag{3}$$

2.2 A Special Case: CreditMetrics with Integrated Correlated Interest Rate Risk

As a special case of the general integrated market and credit portfolio model described before, in this section the usual CreditMetrics framework is extended by interest rate risk which is correlated with transition risk. Then, this extended framework is applied to a homogeneous credit portfolio consisting of N zero coupon bonds with identical face value F and maturity date T issued by N different corporates. This specification of the general integrated model will be used in the numerical example of section 4.

It is assumed that the return X_n on firm n's assets can be described by a normally distributed random variable, which is – without loss of generality – standardized:

$$X_n = \sqrt{\rho_V - \rho_{r,V}^2}\, Z + \rho_{r,V} X_r + \sqrt{1-\rho_V}\,\varepsilon_n \quad (\rho_{r,V}^2 \le \rho_V ,\ n \in \{1,\ldots,N\}), \tag{4}$$

where $Z, X_r, \varepsilon_1,\ldots,\varepsilon_N$ are mutually independent standard normally distributed stochastic variables. The stochastic variables Z and X_r represent systematic credit risk, whereas the ε_n's stand for idiosyncratic credit risk.

The risk-free short rate is modeled for simplicity as a mean-reverting Ornstein-Uhlenbeck process introduced already by *Vasicek* (1977):

$$dr(t) = \kappa(\theta - r(t))\,dt + \sigma_r dW_r(t) , \tag{5}$$

where $\kappa, \theta, \sigma_r \in \mathbb{R}_+$ are positive constants and $W_r(t)$ is a standard Brownian motion under P. The solution of the stochastic differential equation (5) is:

$$r(t) = \underbrace{\theta + (r(0)-\theta)e^{-\kappa t}}_{=E^P[r(t)]} + \sqrt{\frac{\sigma_r^2}{2\kappa}\left(1-e^{-2\kappa t}\right)}X_r , \tag{6}$$

where $X_r \sim N(0,1)$ enters the definition (4) of the firms' asset returns. As it can be easily seen, the definition (4) of the asset returns implies that all pairs of asset returns exhibit a correlation parameter of ρ_V and that the asset returns X_n and the interest rate factor X_r (and hence the short rate $r(H)$) are correlated with parameter $\rho_{r,V}$. It is assumed that the correlation ρ_V between each pair of asset returns as well as the correlation $\rho_{r,V}$ between each asset return and the risk-free short rate are identical.

As in the CreditMetrics methodology, the rating η_H^n of the N obligors at the risk horizon H is simulated by the N-variate normally distributed random vector $X = (X_1, \ldots, X_N)$, whose components exhibit means zero, variances one and equal pairwise correlations ρ_V. An obligor n with current rating i is assumed to be in rating class k at the risk horizon if the realization of X_n lies between two thresholds R_{k+1}^i and R_k^i with $R_{k+1}^i < R_k^i$. The thresholds R_k^i are derived from an one-year transition matrix $Q = (q_{ik})_{1 \le i \le K-1, 1 \le k \le K}$, whose elements q_{ik} specify the unconditional probability that an obligor migrates from the rating grade i to the rating grade k within one year. The thresholds R_k^i ($1 \le i \le K-1,\ 2 \le k \le K$) are computed by ensuring that the probability for the realization of a standardized normally distributed random variable X_n to be in the interval $[R_{k+1}^i, R_k^i]$ coincides with the probability q_{ik} from the migration matrix:

$$R_k^i = \Phi^{-1}\left(\sum_{l=k}^{K} q_{il}\right), \tag{7}$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the cumulative density function of the standard normal distribution. For details concerning this procedure see *Gupton*, *Finger* and *Bhatia* (1997, pp. 85).

The price of a zero coupon bond at the risk horizon H, whose issuer n has not already defaulted until H and exhibits the rating $\eta_H^n \in \{1, \ldots, K-1\}$, is given by:

$$v(X_r, \eta_H^n, H, T) = Fe^{-\left(R(X_r, H, T) + S_{\eta_H^n}(H,T)\right)(T-H)}, \tag{8}$$

where $R(X_r, H, T)$ denotes the stochastic risk-free spot yield for the time interval $[H, T]$, and $S_{\eta_H^n}(H,T)$ is the non-stochastic credit spread of the rating class η_H^n for the time interval $[H, T]$. In the *Vasicek* model the stochastic risk-free spot yield $R(X_r, H, T)$ can easily be calculated in closed-form and is a linear function of the risk factor X_r appearing in (6).

If the issuer n of a zero coupon bond has already defaulted ($\eta_H^n = K$) before the risk horizon H, the value of the bond is set equal to a constant fraction $\delta \in [0,1]$ of the value $p(X_r, H, T)$ of a risk-free but otherwise identical zero coupon bond:

$$v(X_r, K, H, T) = \delta \cdot p(X_r, H, T). \tag{9}$$

The value $\Pi(H)$ of the entire portfolio of defaultable zero coupon bonds at the risk horizon H is:

$$\Pi(H)=\sum_{n=1}^{N}\sum_{k=1}^{K-1}v(X_r,k,H,T)\cdot 1_{\{\eta_H^n=k\}}+\delta\cdot p(X_r,H,T)\cdot 1_{\{\eta_H^n=K\}} \tag{10}$$

where the indicator function $1_{\{\eta_H^n=k\}}$ is one if obligor n is in the rating class k at H and zero otherwise.

The probability of migrating from rating class i to $k\in\{2,\ldots,K-1\}$ until the risk horizon H, conditional on the realizations of the systematic credit risk factors Z and X_r, is given by

$$\begin{aligned} f_{i,k}(z,x_r) &:= P\left(\eta_H^n=k\middle|\eta_0^n=i,Z=z,X_r=x_r\right)=P\left(R_{k+1}^i<X_n\le R_k^i\middle|Z=z,X_r=x_r\right)\\ &=\Phi\left(\frac{R_k^i-\sqrt{\rho_V-\rho_{r,V}^2}\,z-\rho_{r,V}x_r}{\sqrt{1-\rho_V}}\right)-\Phi\left(\frac{R_{k+1}^i-\sqrt{\rho_V-\rho_{r,V}^2}\,z-\rho_{r,V}x_r}{\sqrt{1-\rho_V}}\right). \end{aligned} \tag{11}$$

The conditional default probability is

$$f_{i,K}(z,x_r):=P\left(X_n\le R_K^i\middle|Z=z,X_r=x_r\right)=\Phi\left(\frac{R_K^i-\sqrt{\rho_V-\rho_{r,V}^2}\,z-\rho_{r,V}x_r}{\sqrt{1-\rho_V}}\right), \tag{12}$$

and the conditional probability of being in the best rating class 1 equals

$$f_{i,1}(z,x_r):=P\left(X_n>R_2^i\middle|Z=z,X_r=x_r\right)=1-\Phi\left(\frac{R_2^i-\sqrt{\rho_V-\rho_{r,V}^2}\,z-\rho_{r,V}x_r}{\sqrt{1-\rho_V}}\right). \tag{13}$$

3 Importance Sampling Techniques for the General Approach

As tail events relevant for calculating Value-at-Risk or expected shortfall corresponding to high confidence levels are rare, usually a large number of Monte Carlo simulation runs is needed for computing these risk measures with sufficient accuracy. In this section, importance sampling (IS) is presented as a method to reduce the variance of the Monte Carlo estimators of these risk measures. The usage of this method leads to an improved convergence (in probabilistic terms) of the risk measure estimators when increasing the number of simulation runs so that less simulation runs are necessary to achieve a required accuracy.

3.1 General Remarks about IS

IS attempts to change the original probability measure in such a way that „important“ scenarios of a simulation get more weight so that they occur more frequently and the sampling efficiency is increased. In the context of credit risk modelling „important“ scenarios are those in which the portfolio loss is large. In order to make the central idea of IS, namely the change of measure, more clearly consider the following problem of estimating the mean of some function $h(\cdot)$ of a real-valued random variable X with probability density $f(x)$:[4]

$$\mu_{h(X)} = E^P\left[h(X)\right] = \int_{-\infty}^{\infty} h(x) f(x) dx, \tag{14}$$

where P is the original probability measure. Having simulated D independent draws $X_1,\ldots,X_D$ of the random variable X under P, the ordinary Monte Carlo (MC) estimator for this mean is:

$$\hat{\mu}_{h(X)} = \frac{1}{D}\sum_{d=1}^{D} h(X_d). \tag{15}$$

Next, assuming that the function $g(x)$ is any other probability density on $\mathbb{R}$, which is positive whenever $f(x)$ is positive[5], the mean $\mu_{h(X)}$ can alternatively represented by:

$$\mu_{h(X)} \equiv E^P\left[h(X)\right] = \int_{-\infty}^{\infty} h(x) f(x) dx = \int_{-\infty}^{\infty} h(x)\frac{f(x)}{g(x)} g(x) dx = E^{\tilde{P}}\left[h(x)\frac{f(x)}{g(x)}\right], \tag{16}$$

where $\tilde{P}$ is some new probability measure induced by the density $g(x)$. The quotient in brackets is called the likelihood ratio or the Radon-Nikodým derivative of P with respect to $\tilde{P}$. Using this new probability measure the IS estimator of $\mu_{h(X)}$ is:

$$\hat{\mu}^{\tilde{P}}_{h(X)} = \frac{1}{D}\sum_{d=1}^{D} h(X_d)\frac{f(X_d)}{g(X_d)}, \tag{17}$$

when the independent realizations $X_1,\ldots,X_D$ of the random variable X are sampled under $\tilde{P}$. As we have

$$E^{\tilde{P}}\left[\hat{\mu}^{\tilde{P}}_{h(X)}\right] = \frac{1}{D}\sum_{d=1}^{D} E^{\tilde{P}}\left[h(X_d)\frac{f(X_d)}{g(X_d)}\right] = E^{\tilde{P}}\left[h(X)\frac{f(X)}{g(X)}\right] = E^P\left[h(X)\right] = \mu_{h(X)},$$

[4] See for the following remarks *Glasserman* (2004, pp. 255).

[5] This means that the original probability measure P is absolutely continuous with respect to the new probability measure $\tilde{P}$ which is induced by the density $g(x)$.

the IS estimator $\hat{\mu}_{h(X)}^{\tilde{P}}$ is an unbiased estimator of $\mu_{h(X)}$. The success of an IS algorithm depends on the smart choice of the density $g(x)$, which should decrease the variance of the estimator $\hat{\mu}_{h(X)}^{\tilde{P}}$,

$$Var^{\tilde{P}}\left(\hat{\mu}_{h(X)}^{\tilde{P}}\right)=E^{\tilde{P}}\left[\left(h(X)\frac{f(X)}{g(X)}\right)^2\right]-\left(E^P\left[h(X)\right]\right)^2, \tag{18}$$

compared to that one of the ordinary MC estimator,

$$Var^{P}\left(\hat{\mu}_{h(X)}\right)=E^{P}\left[\left(h(X)\right)^2\right]-\left(E^P\left[h(X)\right]\right)^2. \tag{19}$$

(18) and (19) show that $Var^{\tilde{P}}\left(\hat{\mu}_{h(X)}^{\tilde{P}}\right)<Var^{P}\left(\hat{\mu}_{h(X)}\right)$ is true if and only if the second moment of $\hat{\mu}_{h(X)}^{\tilde{P}}$ is smaller than the second moment of $\hat{\mu}_{h(X)}$. How can we find a density $g(x)$ which decreases the second moment of the estimator $\hat{\mu}_{h(X)}^{\tilde{P}}$ compared that one of $\hat{\mu}_{h(X)}$? In order to get an intuition for finding an adequate density $g(x)$ assume that $h(\cdot)\geq 0$ from which $h(x)f(x)\geq 0$ follows because $f(x)$ is a probability density. Thus, the product $h(x)f(x)$ could also be normalized in order to represent a probability density:

$$g(x):=\frac{h(x)f(x)}{\int_{-\infty}^{\infty}h(x)f(x)dx}, \tag{20}$$

from which

$$\int_{-\infty}^{\infty}h(x)f(x)dx=\frac{h(x)f(x)}{g(x)}$$

and hence

$$const=\int_{-\infty}^{\infty}h(x)f(x)dx=\frac{h(X_d)f(X_d)}{g(X_d)}\quad \forall\, d\in\{1,\ldots,D\} \tag{21}$$

follows. Consequently, employing the function $g(x)$, as defined in (20), as the denominator in the likelihood ratio $f(X_s)/g(X_s)$ would provide a zero-variance estimator $\hat{\mu}_{h(X)}^{\tilde{P}}$ in (17). Unfortunately, the normalizing integral in (20) is just $E^P\left[h(X)\right]$, the term we are looking for, so that we can not determine the optimal (in the sense of variance minimizing) density $g(x)$ according to (20). However, these considerations indicate how an effective IS strategy might look like: Measuring the importance of a realization of the random variable $h(X)$ by the product $h(x)f(x)$ and ensuring that – in this sense – important realizations are sampled more frequently by sampling according to a density $g(x)$ which is proportional to the product $h(x)f(x)$.

3.2 Application of IS to the General Integrated Market and Credit Portfolio Model

In this section the IS technique is applied to the general integrated market and credit portfolio model described in section 2.1. As in *Glasserman* and *Li* (2003a) a two step-IS procedure is applied. First, the conditional transition probabilities $P\left(\eta_H^n = k \middle| \eta_0^n = i, Z_1 = z_1, \ldots, Z_C = z_C\right)$ $= f_{n,i,k}(z_1,\ldots,z_C)$ are modified in order to make defaults and downgrades more probable. Afterwards, the means of the systematic credit risk factors $Z_1,\ldots,Z_C$ and the market risk factors $X_1,\ldots,X_M$, respectively, are shifted in order to make high credit portfolio losses more likely.

For the first step, let us assume that the realizations of the systematic credit risk factors $\mathrm{Z} = (Z_1,\ldots,Z_C)^T$ and the market risk factors $\mathrm{X} = (X_1,\ldots,X_M)^T$ are given. Conditional on the realizations of these random variables, the values of all instruments $p_n(\eta_n^H;\mathrm{X};\mathrm{P_n})$ ($n \in \{1,\ldots,N\}$) are independent.[6] Introducing some new transition probabilities $\tilde{P}_\theta\left(\eta_H^n = k \middle| \eta_0^n = i, \mathrm{Z}, \mathrm{X}\right) = h_{n,i,k}(\mathrm{Z},\mathrm{X})$, we can write the probability that the credit portfolio loss at the risk horizon, defined as[7]

$$L(H) := \sum_{n=1}^{N} L_n(H) = \sum_{n=1}^{N}\left(p_n(\eta_n^0; E^P[\mathrm{X}];\mathrm{P_n}) - p_n(\eta_n^H;\mathrm{X};\mathrm{P_n})\right), \tag{22}$$

is larger than some threshold y in the following way:

$$P\left(L(H) > y \middle| \mathrm{Z},\mathrm{X}\right) = E^{\tilde{P}_\theta}\left[1_{\{L(H)>y\}} \prod_{n=1}^{N}\prod_{k=1}^{K}\left(\frac{f_{n,\eta_n^0,k}(\mathrm{Z})}{h_{n,\eta_n^0,k}(\mathrm{Z},\mathrm{X})}\right)^{1_{\{\eta_n^H = k\}}} \middle| \mathrm{Z},\mathrm{X}\right], \tag{23}$$

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6 Of course, for this conditional independence assumption to be fulfilled, it has to be assumed that also the recovery rates of defaulted obligors are (conditional) independent. For the ease of exposition, in the following, it is assumed that the recovery rate is a deterministic function of the systematic risk factors Z and X so that – conditional on the realizations of these systematic risk factors – the recovery rate is a constant. If one also wants to integrate idiosyncratic recovery rate risk, the recovery rates δ_n would have to be considered as conditioning variables, for example in the definition of the new conditional transition probabilities or the conditional cumulant generating function.

7 Other specifications of the portfolio loss variable (22) are imaginable: For example, instead of $p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})$, the terms $E^P\left[p_n(\eta_n^H;\mathrm{X};\mathrm{P_n})\right]$ or $p_n(\eta_n^0;\mathrm{X}^0;\mathrm{P_n})$, where $\mathrm{X}^0 = (X_1^0,\ldots,X_M^0)^T$ ($m \in \{1,\ldots,M\}$) denotes the current value of the market risk factors in $t = 0$, could be employed.

where $E^{\tilde{P}_\theta}[\,\cdot\,]$ is the expectation operator under the new probability measure $\tilde{P}_\theta$, and the product inside the expectation is the likelihood ratio which relates the original conditional transition probabilities to the new ones. From (23) follows that

$$\frac{1}{D}\sum_{d=1}^{D} 1_{\{L(H)^{(d)}>y\}} \prod_{n=1}^{N}\prod_{k=1}^{K}\left(\frac{f_{n,\eta_n^0,k}(\mathrm{Z})}{h_{n,\eta_n^0,k}(\mathrm{Z},\mathrm{X})}\right)^{1_{\{(\eta_n^H)^{(d)}=k\}}} \tag{24}$$

is an unbiased estimator for $P\left(L(H)>y\,\middle|\,\mathrm{Z},\mathrm{X}\right)$ when the transitions are sampled under the new measure $\tilde{P}_\theta$ and D is the number of samples. In order to make defaults and downgrades more probable, the following definition for the new transition probabilities $h_{n,i,k}(\mathrm{Z},\mathrm{X})$ is used, which is motivated by the exponential twist modification of the default probabilities employed by *Glasserman* and *Li* (2003a) and in a similar way by *Merino* and *Nyfeler* (2004) in the context of pure default mode credit portfolio models:

$$h_{n,i,k}(\mathrm{Z},\mathrm{X}) := \frac{e^{\theta\left(p_n(i;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{X};\mathrm{P_n})\right)} f_{n,i,k}(\mathrm{Z})}{\sum_{k=1}^{K} e^{\theta\left(p_n(i;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{X};\mathrm{P_n})\right)} f_{n,i,k}(\mathrm{Z})}. \tag{25}$$

For $\theta>0$ and $L_n(H)=\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(\eta_n^H;\mathrm{X};\mathrm{P_n})\right)>0$ the transition probabilities are increased, whereas for $\theta>0$ and $L_n(H)<0$ they are diminished. The absolute increase or decrease of the probabilities is larger the higher the individual losses or gains of obligor n's instrument incurred by the respective rating change. For $\theta=0$ the original transition probabilities are not altered. Hence, in general, the downgrade probabilities are increased and the upgrade probabilities are decreased. However, due to the integration of market risk, it is also possible that for example a downgrade probability is decreased, namely in a specific scenario in which a decrease of an instrument's value caused by a downgrade of the issuer is overcompensated by a value increase caused by a movement in the market risk factors. As it can be easily seen, (25) ensures that for all n and i the sum of the new transition probabilities over $k\in\{1,\ldots,K\}$ equals one. For the likelihood ratio in (23) we have the following identity:

$$\prod_{n=1}^{N}\prod_{k=1}^{K}\left(\frac{f_{n,\eta_n^0,k}(\mathrm{Z})}{h_{n,\eta_n^0,k}(\mathrm{Z},\mathrm{X})}\right)^{1_{\{\eta_n^H=k\}}} = e^{-\theta L(H)+\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)} = e^{-\theta\sum_{n=1}^{N}\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(\eta_n^H;\mathrm{X};\mathrm{P_n})\right)+\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)}, \tag{26}$$

where $\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)$ is the (conditional) cumulant generating function of the credit portfolio loss $L(H)$, which is the logarithm of the (conditional) moment generating function:

$$\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta) := \ln\left(E^P\left[e^{\theta L(H)}\,\middle|\,\mathrm{Z},\mathrm{X}\right]\right)=\sum_{n=1}^{N}\ln\left(\sum_{k=1}^{K} e^{\theta\left(p_n(\eta_n^0;E^P[X];\mathrm{P_n})-p_n(k;\mathrm{X};\mathrm{P_n})\right)} f_{n,\eta_n^0,k}(\mathrm{Z})\right). \tag{27}$$

The representation (23) of the excess probability $P\left(L(H)>y \middle| \mathrm{Z},\mathrm{X}\right)$ is useless as long as an adequate parameter $\theta \in \mathbb{R}_+$ is not known. Usually, this parameter is chosen in order to minimize the variance of the estimator (24) under the new probability measure $\tilde{P}_\theta$, which is given by:

$$Var^{\tilde{P}_\theta}\left(1_{\{L(H)>y\}}\prod_{n=1}^{N}\prod_{k=1}^{K}\left(\frac{f_{n,\eta_n^0,k}(\mathrm{Z})}{h_{n,\eta_n^0,k}(\mathrm{Z},\mathrm{X})}\right)^{1_{\{\eta_n^H=k\}}}\middle|\mathrm{Z},\mathrm{X}\right)$$

$$=E^{\tilde{P}_\theta}\left[1_{\{L(H)>y\}}e^{-2\theta L(H)+2\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)}\middle|\mathrm{Z},\mathrm{X}\right]-\left(\underbrace{E^{\tilde{P}_\theta}\left[1_{\{L(H)>y\}}e^{-\theta L(H)+\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)}\middle|\mathrm{Z},\mathrm{X}\right]}_{=P\left(L(H)>y|\mathrm{Z},\mathrm{X}\right)}\right)^2. \tag{28}$$

Thus, as $P\left(L(H)>y \middle| \mathrm{Z},\mathrm{X}\right)$ is independent from θ, minimizing the variance of the estimator (24) is equivalent to minimizing the second moment of this estimator. However, as the random variable, whose second moment is intended to be minimized, as well as the respective probabilities depend on the unknown parameter θ, finding the optimal parameter value is complicated. That is why the same 'trick' as already used by *Glasserman* and *Li* (2003a) and *Merino* and *Nyfeler* (2004) is employed here: Instead of minimizing the second moment of the estimator (24) the following upper boundary of the second moment is minimized by an appropriate choice of the parameter θ:

$$E^{\tilde{P}_\theta}\left[1_{\{L(H)>y\}}e^{-2\theta L(H)+2\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)}\middle|\mathrm{Z},\mathrm{X}\right]=e^{2\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)}E^{\tilde{P}_\theta}\left[1_{\{L(H)>y\}}e^{-2\theta L(H)}\middle|\mathrm{Z},\mathrm{X}\right]$$

$$\underset{\theta\geq 0}{\leq} e^{2\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)}E^{\tilde{P}_\theta}\left[e^{-2\theta y}\middle|\mathrm{Z},\mathrm{X}\right]=e^{-2(\theta y-\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta))}. \tag{29}$$

As

$$\min_{\theta\geq 0} e^{-2(\theta y-\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta))} \tag{30}$$

is equivalent to

$$\max_{\theta\geq 0} \theta y-\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta) \tag{31}$$

and the (conditional) cumulant generating function $\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta)$ is strictly convex in θ with $\psi_{L(H)|\mathrm{Z},\mathrm{X}}(0)=0$,[8] the – in the above sense – optimal parameter θ is given by:

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8 See *Glasserman* (2004, p. 261).

$$\theta_y = \begin{cases} \text{unique solution to } \dfrac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right) = y \text{ for } y > \dfrac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right)\Big|_{\theta=0}, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } y \le \dfrac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right)\Big|_{\theta=0}. \end{cases} \tag{32}$$

The first derivative of the (conditional) cumulant generating function is equal to:

$$\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right) = \frac{\partial}{\partial\theta}\left(\sum_{n=1}^{N}\ln\left(\sum_{k=1}^{K} e^{\theta\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n}) - p_n(k;\mathrm{X};\mathrm{P_n})\right)} f_{n,\eta_n^0,k}(\mathrm{Z})\right)\right)$$

$$= \sum_{n=1}^{N} \frac{\sum_{k=1}^{K}\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n}) - p_n(k;\mathrm{X};\mathrm{P_n})\right) e^{\theta\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n}) - p_n(k;\mathrm{X};\mathrm{P_n})\right)} f_{n,\eta_n^0,k}(\mathrm{Z})}{\sum_{k=1}^{K} e^{\theta\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n}) - p_n(k;\mathrm{X};\mathrm{P_n})\right)} f_{n,\eta_n^0,k}(\mathrm{Z})}. \tag{33}$$

From the (conditional) cumulant generating function $\psi_{L(H)|\mathrm{Z,X}}(\theta)$ important informations about the probability measures $\tilde{P}_\theta$ can be read off. In particular, the first derivative of $\psi_{L(H)|\mathrm{Z,X}}(\theta)$ with respect to θ corresponds to the mean of the random variable $L(H)\big|_{\mathrm{Z,X}}$ under $\tilde{P}_\theta$:[9]

$$\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right) = E^{\tilde{P}_\theta}\left[L(H)\big|\mathrm{Z,X}\right]. \tag{34}$$

Thus, as $P = \tilde{P}_{\theta=0}$, the first case in (32) corresponds to

$$y > \frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right)\Big|_{\theta=0} = E^{\tilde{P}_{\theta=0}}\left[L(H)\big|\mathrm{Z,X}\right] = E^{P}\left[L(H)\big|\mathrm{Z,X}\right], \tag{35}$$

and with the optimal choice of the parameter θ_y we have:

$$y = \frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right)\Big|_{\theta=\theta_y} = E^{\tilde{P}_{\theta_y}}\left[L(H)\big|\mathrm{Z,X}\right]. \tag{36}$$

This means that under the 'optimal' importance sampling measure $\tilde{P}_{\theta_y}$ the mean of the (conditional) credit portfolio loss equals some value y in the upper tail of the random variable $L(H)\big|_{\mathrm{Z,X}}$. Thus, events which were rare under the original measure P are expected events under the new measure $\tilde{P}_{\theta_y}$.

9 See *Glasserman* (2004, p. 261).

For the second step, the means of the systematic credit risk factors $Z_1,\dots,Z_C$ and the market risk factors $X_1,\dots,X_M$ respectively are shifted in order to make high losses more likely.[10] Fur this purpose, we adapt a technique presented by *Glasserman* and *Li* (2003a) for a pure default mode credit risk model to the general integrated market and credit portfolio model of section 2.1.[11] Previously, this technique has also been used in the context of pricing path-dependent options.[12]

The remaining problem in the second step is to speed up the estimation of the expectation:

$$P\left(L(H)>y\right)=E^P\left[P\left(L(H)>y\middle|\mathrm{Z},\mathrm{X}\right)\right]. \tag{37}$$

As described before in section 3.1, a good IS strategy might be to sample Z and X according to a density which is proportional to the product $P\left(L(H)>y\middle|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}\right)\cdot f(\mathrm{z},\mathrm{x})$, where $f(\mathrm{z},\mathrm{x})$ with $f:\mathbb{R}^C\times\mathbb{R}^M\to\mathbb{R}_+$ is the probability density of the $(C+M)$-dimensional random vector (Z,X) under the original probability measure P. For the ease of exposition, it is assumed that each component of the random vector (Z,X) is standard normally distributed and that the individual components are not correlated.[13] Hence, the function $g(\mathrm{z},\mathrm{x})$ in the denominator of the likelihood ratio in (17) should be proportional to:

$$P\left(L(H)>y\middle|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}\right)e^{-0.5\left(\sum_{c=1}^{C}z_c^2+\sum_{m=1}^{M}x_m^2\right)}. \tag{38}$$

However, the problem in finding the optimal[14] density $g(\mathrm{z},\mathrm{x})$ consists in finding the proportionality constant which would also be the normalization constant making (20) to be a density function. In order to overcome this problem, *Glasserman* and *Li* (2003a) follow a proposal of *Glasserman*, *Heidelberger* and *Shahabuddin* (1999), which was used by these authors in the

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10 In the context of the original CreditMetrics model the idea of this IS strategy has been first, albeit in an informal way, described by *Xiao* (2001).

11 For an alternative technique see *Kalkbrenner*, *Lotter* and *Overbeck* (2004). Within the framework of the default mode CreditMetrics model these authors try to find optimal means of the systematic credit risk factors under the IS distribution by approximating the original inhomogeneous, finite portfolio by a homogeneous, infinitely granular portfolio. Of course, this approximation procedure is not unique. Then, they calculate the mean of the systematic credit risk factor which minimizes the variance of the estimator of the desired risk measure in a one-factor model of that homogeneous, infinitely granular portfolio. Finally, they 'lift' this one-dimensional optimal mean to a M-dimensional mean vector. An approximation of the original portfolio by an infinitely granular, homogeneous portfolio is also used by *Tchistiakov*, *de Smet* and *Hoogbruin* (2004) in order to reduce the variance of the risk measure estimator. However, they employ this approximation as a control variable.

12 See *Glasserman*, *Heidelberger* and *Shahabuddin* (1999).

13 If the joint distribution of the random vector (Z,X) is a multivariate normal distribution, this assumption is without loss of generality, because a set of correlated normally distributed random variables can always be represented by a linear combination of orthogonal standard normally distributed random variables.

14 The density $g(\mathrm{z},\mathrm{x})$ would be optimal if it would provide a zero variance IS estimator (see section 3.1).

context of option pricing: In this approach, the function $g(\mathrm{z},\mathrm{x})$ is assumed to be the density function of a multivariate normal distribution with mean vector $\mu \in \mathbb{R}^{C+M}$ and a covariance matrix equal to the identity matrix $\mathrm{I} \in \mathbb{R}^{(C+M)\times(C+M)}$. The mean vector μ is chosen as the mode of the optimal density, which equals the mode of (38):[15]

$$\mu = \underset{z_1,\dots,z_C,x_1,\dots,x_M \in \mathbb{R}}{\arg\max}\; P\left(L(H) > y \middle| \mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}\right) e^{-0.5\left(\sum_{c=1}^{C} z_c^2 + \sum_{m=1}^{M} x_m^2\right)}$$

$$= \underset{z_1,\dots,z_C,x_1,\dots,x_M \in \mathbb{R}}{\arg\max}\; E^{\tilde{P}_{\theta_y(\mathrm{z},\mathrm{x})}}\left[1_{\{L(H)>y\}} e^{-\theta_y(\mathrm{z},\mathrm{x})L(H)+\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))} \middle| \mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}\right] e^{-0.5\left(\sum_{c=1}^{C} z_c^2 + \sum_{m=1}^{M} x_m^2\right)}. \quad (39)$$

In the above representation the dependence of the parameter θ_y on the realizations $\mathrm{Z}=\mathrm{z}$ and $\mathrm{X}=\mathrm{x}$ is expressed by the notation $\theta_y = \theta_y(\mathrm{z},\mathrm{x})$. In order to simplify the optimization problem (39), a similar approximation as before is used:

$$E^{\tilde{P}_{\theta_y(\mathrm{z},\mathrm{x})}}\left[1_{\{L(H)>y\}} e^{-\theta_y(\mathrm{z},\mathrm{x})L(H)+\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))} \middle| \mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}\right] \le e^{\overbrace{-\theta_y(\mathrm{z},\mathrm{x})y+\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}^{=:F_y(\mathrm{z},\mathrm{x})}}. \quad (40)$$

The conditional probability $P\left(L(H) > y \middle| \mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}\right)$ is substituted by the upper boundary (40) in the optimization problem (39). This yields:

$$\mu = \underset{z_1,\dots,z_C,x_1,\dots,x_M \in \mathbb{R}}{\arg\max}\; F_y(\mathrm{z},\mathrm{x}) - 0.5\left(\sum_{c=1}^{C} z_c^2 + \sum_{m=1}^{M} x_m^2\right). \quad (41)$$

Thus, the mean vector μ of $g(\mathrm{z},\mathrm{x})$ is given by the solution of the following equations:

$$\frac{\partial}{\partial z_c}\left(F_y(\mathrm{z},\mathrm{x}) - 0.5\left(\sum_{c=1}^{C} z_c^2 + \sum_{m=1}^{M} x_m^2\right)\right)$$

$$= -\frac{\partial\theta_y(\mathrm{z},\mathrm{x})}{\partial z_c} y + \frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial z_c} + \frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial\theta_y}\frac{\partial\theta_y(\mathrm{z},\mathrm{x})}{\partial z_c} - z_c = 0 \quad (42)$$

$$(c \in \{1,\dots,C\}),$$

and, analogously,

$$\frac{\partial}{\partial x_m}\left(F_y(\mathrm{z},\mathrm{x}) - 0.5\left(\sum_{c=1}^{C} z_c^2 + \sum_{m=1}^{M} x_m^2\right)\right)$$

$$= -\frac{\partial\theta_y(\mathrm{z},\mathrm{x})}{\partial x_m} y + \frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial x_m} + \frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial\theta_y}\frac{\partial\theta_y(\mathrm{z},\mathrm{x})}{\partial x_m} - x_m = 0 \quad (43)$$

$$(m \in \{1,\dots,M\}).$$

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[15] As the optimal density is proportional to (38), the mode of the optimal density coincides with the mode of (38). If there are several modes, the one which delivers the highest maximum should be chosen. However, in the numerical example of section 4 this problem does not appear.

For $\left\{(\mathrm{z},\mathrm{x})\in\mathbb{R}^{C+M}\,\middle|\, y>E^P\left[L(H)\middle|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}\right]\right\}$, due to (32) and (35), these equations can be simplified to:

$$-\frac{\partial\theta_y(\mathrm{z},\mathrm{x})}{\partial z_c}y+\frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial z_c}+y\frac{\partial\theta_y(\mathrm{z},\mathrm{x})}{\partial z_c}-z_c$$

$$=\frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial z_c}-z_c=0, \tag{44}$$

and

$$\frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial x_m}-x_m=0. \tag{45}$$

For $\left\{(\mathrm{z},\mathrm{x})\in\mathbb{R}^{C+M}\,\middle|\, y\le E^P\left[L(H)\middle|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}\right]\right\}$ (32) and (35) yield the same representation of the necessary conditions for a maximum:

$$-\underbrace{\frac{\partial\theta_y(\mathrm{z},\mathrm{x})}{\partial z_c}}_{\underset{\theta_y=0}{=}0}y+\frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial z_c}+\frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial\theta_y}\underbrace{\frac{\partial\theta_y(\mathrm{z},\mathrm{x})}{\partial z_c}}_{\underset{\theta_y=0}{=}0}-z_c$$

$$=\frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial z_c}-z_c=0, \tag{46}$$

and

$$\frac{\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))}{\partial x_m}-x_m=0. \tag{47}$$

The first derivative of the (conditional) cumulant generating function $\partial\psi_{L(H)|\mathrm{Z}=\mathrm{z},\mathrm{X}=\mathrm{x}}(\theta_y(\mathrm{z},\mathrm{x}))$ with respect to z_c and x_m, respectively, are given by:

$$\frac{\partial}{\partial z_c}\left(\sum_{n=1}^{N}\ln\left(\sum_{k=1}^{K}e^{\theta_y(\mathrm{z},\mathrm{x})\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{x};\mathrm{P_n})\right)}f_{n,\eta_n^0,k}(\mathrm{z})\right)\right)$$

$$=\sum_{n=1}^{N}\frac{\sum_{k=1}^{K}e^{\theta_y(\mathrm{z},\mathrm{x})\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{x};\mathrm{P_n})\right)}\frac{\partial f_{n,\eta_n^0,k}(\mathrm{z})}{\partial z_c}}{\sum_{k=1}^{K}e^{\theta_y(\mathrm{z},\mathrm{x})\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{x};\mathrm{P_n})\right)}f_{n,\eta_n^0,k}(\mathrm{z})}\quad(c\in\{1,\ldots,C\}), \tag{48}$$

and

$$\frac{\partial}{\partial x_m}\left(\sum_{n=1}^{N}\ln\left(\sum_{k=1}^{K}e^{\theta_y(\mathrm{z},\mathrm{x})\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{x};\mathrm{P_n})\right)}f_{n,\eta_n^0,k}(\mathrm{z})\right)\right)$$

$$=\sum_{n=1}^{N}\frac{\sum_{k=1}^{K}-\theta_y(\mathrm{z},\mathrm{x})\dfrac{\partial p_n(k;\mathrm{x};\mathrm{P_n})}{\partial x_m}e^{\theta_y(\mathrm{z},\mathrm{x})\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{x};\mathrm{P_n})\right)}f_{n,\eta_n^0,k}(\mathrm{z})}{\sum_{k=1}^{K}e^{\theta_y(\mathrm{z},\mathrm{x})\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{x};\mathrm{P_n})\right)}f_{n,\eta_n^0,k}(\mathrm{z})}\quad(m\in\{1,\ldots,M\}).\quad(49)$$

If (some of) the market risk factors X have also explanatory power for the original transition probabilities[16], these last M equations have to be modified to:

$$\frac{\partial}{\partial x_m}\left(\sum_{n=1}^{N}\ln\left(\sum_{k=1}^{K}e^{\theta_y(\mathrm{z},\mathrm{x})\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{x};\mathrm{P_n})\right)}f_{n,\eta_n^0,k}(\mathrm{z},\mathrm{x})\right)\right)$$

$$=\sum_{n=1}^{N}\frac{\sum_{k=1}^{K}e^{\theta_y(\mathrm{z},\mathrm{x})\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{x};\mathrm{P_n})\right)}\left(\dfrac{\partial f_{n,\eta_n^0,k}(\mathrm{z},\mathrm{x})}{\partial x_m}-\theta_y(\mathrm{z},\mathrm{x})\dfrac{\partial p_n(k;\mathrm{x};\mathrm{P_n})}{\partial x_m}f_{n,\eta_n^0,k}(\mathrm{z},\mathrm{x})\right)}{\sum_{k=1}^{K}e^{\theta_y(\mathrm{z},\mathrm{x})\left(p_n(\eta_n^0;E^P[\mathrm{X}];\mathrm{P_n})-p_n(k;\mathrm{x};\mathrm{P_n})\right)}f_{n,\eta_n^0,k}(\mathrm{z},\mathrm{x})}.\quad(50)$$

Having determined the mean vector μ of the density $g(\mathrm{z},\mathrm{x})$, the whole IS estimator for the excess probability $P(L(H)>y)$, combining both steps described before, is finally:

$$P(L(H)>y)=E^P\left[P\left(L(H)>y\middle|\mathrm{Z},\mathrm{X}\right)\right]=E^P\left[E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{X})}}\left[1_{\{L(H)>y\}}\prod_{n=1}^{N}\prod_{k=1}^{K}\left(\frac{f_{n,\eta_n^0,k}(\mathrm{Z})}{h_{n,\eta_n^0,k}(\mathrm{Z})}\right)^{1_{\{\eta_n^H=k\}}}\middle|\mathrm{Z},\mathrm{X}\right]\right]$$

$$\approx\frac{1}{D}\sum_{d=1}^{D}1_{\{L(H)^{(d)}>y\}}e^{-\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)})L(H)^{(d)}+\psi_{L(H)|\mathrm{Z}^{(d)},\mathrm{X}^{(d)}}(\theta_y(\mathrm{Z}^{(d)},\mathrm{X}^{(d)}))}e^{-\sum_{c=1}^{C}\left(Z_c^{(d)}\mu_c-0.5\mu_c^2\right)-\sum_{m=1}^{M}\left(X_m^{(d)}\mu_m-0.5\mu_m^2\right)},\quad(51)$$

where $(\mathrm{Z},\mathrm{X})\sim N(\mu,\mathrm{I})$, D is the number of draws of the systematic risk factors Z and X, $\theta_y(\mathrm{Z},\mathrm{X})$ is given by (32) and – conditional on the realization of (Z,X) – the credit portfolio loss $L(H)$ is sampled according to the modified transition probabilities $h_{n,i,k}(\mathrm{Z},\mathrm{X})$ (see (25)).

When we are not interested in the Value-at-Risk and, hence, in the calculation of excess probabilities and percentiles of the loss variable $L(H)$, but prefer the expected shortfall as a measure of the risk of the credit portfolio, then

$$E^P\left[L(H)\middle|L(H)>y\right]=\frac{1}{1-p}E^P\left[E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{X})}}\left[L(H)\cdot 1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{Z},\mathrm{X})L(H)+\psi_{L(H)|\mathrm{Z},\mathrm{X}}(\theta_y(\mathrm{Z},\mathrm{X}))}\middle|\mathrm{Z},\mathrm{X}\right]\right]$$

has to be estimated. In this case, for determining an appropriate parameter θ, the upper boundary (29) is replaced by:[17]

[16] This is the case in the example of an integrated market and credit portfolio model described in section 2.2, which is also employed for the numerical example in section 4.

$$E^{\tilde{P}_\theta}\left[L(H)^2\cdot 1_{\{L(H)>y\}}e^{-2\theta L(H)+2\psi_{L(H)|\mathrm{Z,X}}(\theta)}\middle|\mathrm{Z,X}\right]\underset{\theta\geq 0}{\leq} e^{-2(\theta y-\psi_{L(H)|\mathrm{Z,X}}(\theta))}E^{\tilde{P}_\theta}\left[L(H)^2\middle|\mathrm{Z,X}\right]$$

$$=e^{-2(\theta y-\psi_{L(H)|\mathrm{Z,X}}(\theta))}\left(\frac{\partial^2}{\partial\theta^2}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right)+\left(\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right)\right)^2\right),$$

and θ is given by the solution of the optimization problem:

$$\theta_y(\mathrm{Z,X})=\underset{\theta\geq 0}{\arg\max}\, e^{-2(\theta y-\psi_{L(H)|\mathrm{Z,X}}(\theta))}\left(\frac{\partial^2}{\partial\theta^2}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right)+\left(\frac{\partial}{\partial\theta}\left(\psi_{L(H)|\mathrm{Z,X}}(\theta)\right)\right)^2\right),$$

which has to be solved for each realization of the systematic risk factors. Similarly, the IS mean $\mu\in\mathbb{R}^{\mathrm{C+M}}$ of the systematic risk factors is given as the solution of the optimization problem:

$$\mu=\underset{z_1,\ldots,z_C,x_1,\ldots,x_M\in\mathbb{R}}{\arg\max}\, E^{\tilde{P}_{\theta_y(\mathrm{z,x})}}\left[L(H)\cdot 1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{z,x})L(H)+\psi_{L(H)|\mathrm{Z=z,X=x}}(\theta_y(\mathrm{z,x}))}\middle|\mathrm{Z=z,X=x}\right]e^{-0.5\left(\sum_{c=1}^{C}z_c^2+\sum_{m=1}^{M}x_m^2\right)}.$$

As these modified optimization problem are more complex and, hence, it is more time-consuming to solve them, in the numerical example of section 4 it is analyzed which variance reduction can be achieved when for expected shortfall estimations simply the same parameters θ and μ are used as before for the computation of excess probabilities. Because of the relation $y=E^{\tilde{P}_{\theta_y(\mathrm{Z,X})}}\left[L(H)\middle|\mathrm{Z,X}\right]$ (see 36)), it is obvious that this procedure can not be optimal; the IS approach for the estimation of the expectation of all realizations of the loss variable $L(H)$ which are larger than some threshold y would be certainly more effective when the mean of the (conditional) credit portfolio loss is larger than y under the IS measure $\tilde{P}_{\theta_y(\mathrm{Z,X})}$. However, the numerical experiments in section 4 show that a significant variance reduction can be achieved even with the use of the suboptimal parameters θ and μ.

3.3 Modification: Insertion of a Third Step

As we deal here with an integrated market and credit portfolio model, it might suggest itself to employ also IS techniques originally developed for pure market risk portfolio models and to combine these with those techniques originally developed for pure credit risk portfolio models. This is what we want to try next. As a result we get a three step-IS algorithm, where the additional step can either inserted before or after the second step described in section 3.2. Instead of considering the optimization problem (39) for finding variance reducing means of

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[17] For the last transformation it is used that the second derivative of the CGF evaluated at θ equals the variance of the random variable under the probability measure $\tilde{P}_\theta$.

both kinds of systematic risk factors, now, this procedure is only carried out for the systematic credit risk factors $Z_1,\ldots,Z_C$, whereas the IS distribution for the market risk factors $X_1,\ldots,X_M$ is determined in an intermediate step. For finding the IS distribution of the market risk factors it is assumed that the credit portfolio is default risk-free and that all obligors remain in their initial rating class until the risk horizon. With this assumption the approach of *Glasserman*, *Heidelberger* and *Shahabuddin* (2000) developed for pure market risk portfolio models can be applied. Their method employs a delta-gamma approximation of the portfolio value at the risk horizon in order to find a variance reducing IS distribution for the market risk factors, whose changes over the risk horizon are assumed to be multivariate normally distributed.[18] It has to be stressed that the assumption that all obligors remain in their initial rating class is only used for finding an effective IS distribution for the market risk factors, but, of course, not as a real approximation of the credit portfolio value.

The random variable representing the credit portfolio loss which is only due to movements in the market risk factors over the risk horizon is defined as:[19]

$$L^{wtr}(\mathrm{X},H) := \sum_{n=1}^{N} L_n^{wtr}(\mathrm{X},H) = \sum_{n=1}^{N}\Big(p_n(\eta_n^0; E^P[\mathrm{X}]; \mathrm{P_n}) - p_n(\eta_n^0; \mathrm{X}; \mathrm{P_n})\Big), \tag{52}$$

where the upper index wtr indicates that this is the loss *w*ithout *t*ransition *r*isk. For this random variable a specific quadratic approximation, the so-called delta-gamma approximation, is introduced:

$$\begin{aligned} &L^{wtr}(\mathrm{X},H) \approx L^{wtr,\Delta,\Gamma}(\mathrm{X},H) \\ &= L^{wtr}(E^P[\mathrm{X}],H) + \delta^{\mathrm{T}} \underbrace{(\mathrm{X}-E^P[\mathrm{X}])}_{=:\Delta\mathrm{X}} + 0.5 \underbrace{(\mathrm{X}-E^P[\mathrm{X}])^{\mathrm{T}}}_{=:\Delta\mathrm{X}^{\mathrm{T}}} \Gamma \underbrace{(\mathrm{X}-E^P[\mathrm{X}])}_{=:\Delta\mathrm{X}}, \end{aligned} \tag{53}$$

where the column vector $\Delta\mathrm{X} \sim N(0,\Sigma_{\mathrm{X}})$ is the multivariate normally distributed[20] difference between the realized values of the market risk factors at H and their expected values[21], the

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[18] *Glasserman*, *Heidelberger* and *Shahabuddin* (2000) show that when the delta-gamma approximation is exact, their IS technique is 'asymptotically optimal' for estimating exceedance probabilities $P(L > y)$ for large y, where 'asymptotic optimality' means that the second moment of the IS estimator for $P(L > y)$ decreases at the fastest possible exponential rate as y increases. Besides, *Glasserman*, *Heidelberger* and *Shahabuddin* (2000) use stratified sampling as an additional variance reduction technique. For an application of the delta-gamma approximation as a control variate see *Glasserman* (2004, pp. 493). For IS combined with stratified sampling under the assumption of multivariate t-distributed risk factors see *Glasserman*, *Heidelberger* and *Shahabuddin* (2002).

[19] Note that both terms depend on the current rating η_n^0.

[20] In spite of having assumed $\mathrm{X} \sim N(0,\mathrm{I})$ in the previous section, we use the above multivariate normal assumption for the market risk factors in order to follow the original presentation of this approach more closely.

[21] In the case that the market risk factors are modelled by a lognormal distribution, a quadratic approximation of the kind (53) is still possible. In this case the market risk factors have the representation $Y_m = Y_m^0 e^{c_m + d_m X_m}$

column vector $\delta = (\delta_m)_{1 \le m \le M}$ contains the first derivatives of $L^{wtr}(\mathrm{X}, H)$ with respect to the market risk factors:

$$\delta_m = \left. \frac{\partial L^{wtr}(\mathrm{X}, H)}{\partial X_m} \right|_{\mathrm{X}=E^P[\mathrm{X}]} = \sum_{n=1}^{N} - \left. \frac{\partial p_n(\eta_n^0; \mathrm{X}; \mathrm{P_n})}{\partial X_m} \right|_{\mathrm{X}=E^P[\mathrm{X}]} \quad (m \in \{1, \ldots, M\}), \tag{54}$$

and the matrix $\Gamma = (\Gamma_{m,n})_{1 \le m,n \le M}$ is the Hessian matrix with the second derivatives of $L^{wtr}(\mathrm{X}, H)$ with respect to the market risk factors:

$$\Gamma_{m,n} = \left. \frac{\partial^2 L^{wtr}(\mathrm{X}, H)}{\partial X_m \partial X_n} \right|_{\mathrm{X}=E^P[\mathrm{X}]} = \sum_{n=1}^{N} - \left. \frac{\partial^2 p_n(\eta_n^0; \mathrm{X}; \mathrm{P_n})}{\partial X_m \partial X_n} \right|_{\mathrm{X}=E^P[\mathrm{X}]} \quad (m, n \in \{1, \ldots, M\}). \tag{55}$$

Hence, (53) is just a second order Taylor series expansion of the credit portfolio loss $L^{wtr}(\mathrm{X}, H)$ around the expected market risk factors at the risk horizon.

Next, a more convenient expression for the quadratic approximation (53) is derived.[22] For this, let $\tilde{\mathrm{C}} \in \mathbb{R}^{M \times M}$ be a quadratic matrix which fulfils $\tilde{\mathrm{C}}\tilde{\mathrm{C}}^{\mathrm{T}} = \Sigma_{\mathrm{X}}$ (the matrix $\tilde{C}$ can be obtained for example from the Cholesky decomposition of Σ_{X}). Then, the matrix $0.5\tilde{\mathrm{C}}^{\mathrm{T}}\Gamma\tilde{\mathrm{C}}$ is diagonalized which is always possible because this is a real-valued symmetric matrix. Thus, the following representation is possible:

$$0.5\tilde{\mathrm{C}}^{\mathrm{T}}\Gamma\tilde{\mathrm{C}} = \mathrm{U}\Lambda\mathrm{U}^{\mathrm{T}}, \tag{56}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_M \end{pmatrix} \tag{57}$$

is a diagonal matrix containing the eigenvalues of $0.5\tilde{\mathrm{C}}^{\mathrm{T}}\Gamma\tilde{\mathrm{C}}$ and U is an orthogonal matrix whose columns are the eigenvectors of $0.5\tilde{\mathrm{C}}^{\mathrm{T}}\Gamma\tilde{\mathrm{C}}$. Defining $\mathrm{C} = \tilde{\mathrm{C}}\mathrm{U}$ and $\mathrm{S} = (S_1, \ldots, S_M)^{\mathrm{T}}$ as a vector of independent standard normally distributed random variables, $\Delta\mathrm{X} = \mathrm{CS}$ has a $N(0, \Sigma_{\mathrm{X}})$ distribution because

$$\mathrm{CC}^{\mathrm{T}} = \tilde{\mathrm{C}}\mathrm{U}(\tilde{\mathrm{C}}\mathrm{U})^{\mathrm{T}} = \tilde{\mathrm{C}}\mathrm{U}\mathrm{U}^{\mathrm{T}}\tilde{\mathrm{C}}^{\mathrm{T}} = \tilde{\mathrm{C}}\tilde{\mathrm{C}}^{\mathrm{T}} = \Sigma_{\mathrm{X}}. \tag{58}$$

Finally, observing that

$$0.5\mathrm{C}^{\mathrm{T}}\Gamma\mathrm{C} = 0.5(\tilde{\mathrm{C}}\mathrm{U})^{\mathrm{T}}\Gamma(\tilde{\mathrm{C}}\mathrm{U}) = \mathrm{U}^{\mathrm{T}}\left(0.5\tilde{\mathrm{C}}^{\mathrm{T}}\Gamma\tilde{\mathrm{C}}\right)\mathrm{U} = \mathrm{U}^{\mathrm{T}}\left(\mathrm{U}\Lambda\mathrm{U}^{\mathrm{T}}\right)\mathrm{U} = \Lambda, \tag{59}$$

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with $c_m, d_m \in \mathbb{R}$ and $\mathrm{X} = (X_1, \ldots, X_M)^T \sim N(\mu_{\mathrm{X}}, \Sigma_{\mathrm{X}})$ and, applying the chain rule, an approximation of $L^{wtr}(\mathrm{X}, H)$, which is quadratic in X, could still be derived (see *Glasserman*, *Heidelberger* and *Shahabuddin* (2000, p. 1351)).

[22] See *Glasserman* (2004, pp. 486) and *Glasserman*, *Heidelberger* and *Shahabuddin* (2000, p. 1351).

(53) can be written as

$$L^{wtr}(\mathrm{X},H) \approx L^{wtr,\Delta,\Gamma}(\mathrm{X},H) = \underbrace{L^{wtr}(E^{P}[\mathrm{X}],H)}_{=:a} + \delta^{\mathrm{T}}\Delta\mathrm{X} + 0.5\Delta\mathrm{X}^{\mathrm{T}}\Gamma\Delta\mathrm{X}$$

$$= a + \delta^{\mathrm{T}}\mathrm{CS} + 0.5(\mathrm{CS})^{\mathrm{T}}\Gamma\mathrm{CS} = a + \underbrace{\delta^{\mathrm{T}}\mathrm{C}}_{=:\mathrm{b}}\mathrm{S} + \mathrm{S}^{\mathrm{T}}\underbrace{0.5\mathrm{C}^{\mathrm{T}}\Gamma\mathrm{C}}_{=\Lambda}\mathrm{S}$$

$$= a + \sum_{m=1}^{M}\left(b_m S_m + \lambda_m S_m^2\right) =: a + Q(\mathrm{S}). \tag{60}$$

In the next step, the approximation $L^{wtr}(\mathrm{X},H) \approx a + Q(\mathrm{S})$ is used for finding an IS distribution for the market risk factors under which large values of the portfolio loss are generated with an higher probability than under the original distribution. Considering (60) one can see that large values of $a + Q(\mathrm{S})$ are more likely, when[23]

- those random variables S_m for which $b_m > 0$ have a positive mean,
- those random variables S_m for which $b_m < 0$ have a negative mean and
- those random variables S_m for which $\lambda_m > 0$ have a large variance.

Any effective IS distribution for the market risk factors should consider these effects. It is assumed that the IS distribution is multivariate normal again, but with modified mean vector and modified covariance matrix. Then the likelihood ratio $l(\mathrm{S})$ is:

$$l(\mathrm{S}) = \frac{e^{-\frac{1}{2}\mathrm{S}^{\mathrm{T}}\mathrm{S}}}{\det(\Sigma(\theta^{wtr}))^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathrm{S}-\mu(\theta^{wtr}))^{\mathrm{T}}\Sigma(\theta^{wtr})^{-1}(\mathrm{S}-\mu(\theta^{wtr}))}}. \tag{61}$$

The entries of the mean vector $\mu(\theta^{wtr})$ and the covariance matrix $\Sigma(\theta^{wtr})$ are chosen as follows:[24]

$$\mu_m(\theta^{wtr}) = \frac{\theta^{wtr} b_m}{1 - 2\theta^{wtr}\lambda_m} \quad (m \in \{1,\ldots,M\}), \tag{62}$$

$$\sigma_{mn}^2(\theta^{wtr}) = \begin{cases} \dfrac{1}{1-2\theta^{wtr}\lambda_m} & (m = n) \\ 0 & (m \neq n) \end{cases} \quad (m,n \in \{1,\ldots,M\}), \tag{63}$$

where

$$\theta^{wtr} \in \left[0, \frac{1}{\max\limits_{1 \le m \le M} 2\lambda_m}\right) \tag{64}$$

[23] See *Glasserman* (2004, p. 495).
[24] See *Glasserman*, *Heidelberger* and *Shahabuddin* (2000, p. 1352).

is required for the parameter θ^{wtr} if $\max_{1\le m\le M} \lambda_m > 0$. With this choice of the IS distribution the likelihood ratio (61) simplifies to:

$$l(\mathrm{S}) = e^{-\theta^{wtr}Q(\mathrm{S})+\psi_Q(\theta^{wtr})}, \tag{65}$$

where[25]

$$\psi_Q(\theta^{wtr}) \equiv \ln\left(E^P\left[e^{\theta^{wtr}Q(\mathrm{S})}\right]\right) = \frac{1}{2}\sum_{m=1}^{M}\left(\frac{(\theta^{wtr})^2 b_m^2}{1-2\theta^{wtr}\lambda_m} - \ln\left(1-2\theta^{wtr}\lambda_m\right)\right) \tag{66}$$

is the cumulant moment generating function of the random variable $Q(\mathrm{S})$. The identity (65) shows that employing (62) and (63) corresponds to an exponential change of measure for the quadratic form $Q(\mathrm{S})$. With this specific choice of the IS distribution the random variables S_m ($m \in \{1,\ldots,M\}$) still remain independent. Furthermore, the above mentioned aspects which should be reflected by any IS distribution of the market risk factors are indeed considered.

In a final step, the parameter θ^{wtr} has to be determined. For this, the approximation $P(L^{wtr}(H) > y^*) \approx P(a + Q(\mathrm{S}) > y^*)$ is used and a parameter θ^{wtr} which is effective for estimating the probability on the right-hand side is computed, hoping that it is also effective for estimating the probability on the left-hand side.[26] First, as in the first step of the IS technique described in section 3.2, an upper boundary for the second moment of the IS estimator for $P(a + Q(\mathrm{S}) > y^*)$ is determined:

$$E^{\tilde{P}_{\theta^{wtr}}}\left[\left(1_{\{Q(\mathrm{S})>y^*-a\}}l(\mathrm{S})\right)^2\right] = E^{\tilde{P}_{\theta^{wtr}}}\left[1_{\{Q(\mathrm{S})>y^*-a\}}e^{-2\theta^{wtr}Q(\mathrm{S})+2\psi_Q(\theta^{wtr})}\right] \underset{\theta\ge 0}{\le} e^{-2\left(\theta^{wtr}(y^*-a)-\psi_Q(\theta^{wtr})\right)}. \tag{67}$$

As minimizing the second moment of the IS estimator is difficult, instead, θ^{wtr} is chosen in order to minimize the upper boundary (67). Analogously to section 3.2, this yields:

$$\theta_{y^*}^{wtr} = \begin{cases} \text{unique solution to } \dfrac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right) = y^*-a \text{ for } y^*-a > \left.\dfrac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right)\right|_{\theta^{wtr}=0}, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } y^*-a \le \left.\dfrac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right)\right|_{\theta^{wtr}=0}, \end{cases} \tag{68}$$

where the first derivative of the cumulant generating function is given by:

$$\frac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right) = \sum_{m=1}^{M}\left(\frac{\theta^{wtr}b_m^2(1-\theta^{wtr}\lambda_m)}{(1-2\theta^{wtr}\lambda_m)^2} + \frac{\lambda_m}{1-2\theta^{wtr}\lambda_m}\right). \tag{69}$$

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25 See *Glasserman* (2004, p. 487).

26 In the following, we differ between the two figures y and y^*: y is the initial guess of the percentile of the credit portfolio loss distribution which we are looking for, whereas y^* is the initial guess of a percentile of the loss distribution when we only consider market risk but no transition risk. Especially for portfolios with a low credit quality these figures differ significantly, even if the percentiles correspond to the same confidence level.

As in section 3.2, under the IS distribution $\tilde{P}_{\theta_{y^*}^{wtr}}$ the mean of the random variable $a+Q(\mathrm{S})$ is equal to y^*, so that $\{L^{wtr}(H)>y^*\}\approx\{a+Q(\mathrm{S})>y^*\}$ is no longer a rare event under the new sampling distribution:

$$E^{\tilde{P}_{\theta_{y^*}^{wtr}}}\left[Q(\mathrm{S})\right]=\frac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right)\Big|_{\theta^{wtr}=\theta_{y^*}^{wtr}}=y^*-a$$

$$\Leftrightarrow E^{\tilde{P}_{\theta_{y^*}^{wtr}}}\underbrace{\left[a+Q(\mathrm{S})\right]}_{\approx L^{wtr}(H)}=y^*\quad\text{for } y^*-a>\frac{\partial}{\partial\theta^{wtr}}\left(\psi_Q(\theta^{wtr})\right)\Big|_{\theta^{wtr}=0}. \tag{70}$$

3.4 Combination of the Three Steps

Finally, there are several possibilities how to combine the two steps of section 3.2 and the one described in this section in order to build a complete IS estimator for the excess probability $P(L(H)>y)$. These are discussed in the following.

One possibility would be the following approach:

$$P(L(H)>y)=E^P\left[P\left(L(H)>y\,\middle|\,\mathrm{Z},\mathrm{X}\right)\right]$$

$$=E^P\left[E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{CS})}}\left[1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{Z},\mathrm{CS})L(H)+\psi_{L(H)|\mathrm{Z},\mathrm{CS}}(\theta_y(\mathrm{Z},\mathrm{CS}))}\,\middle|\,\mathrm{Z},\mathrm{S}\right]\right]$$

$$=E^P\left[E^{\tilde{P}_{\theta_{y^*}^{wtr}}}\left[E^{\tilde{P}_{\theta_y(\mathrm{Z},\mathrm{CS})}}\left[1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{Z},\mathrm{CS})L(H)+\psi_{L(H)|\mathrm{Z},\mathrm{CS}}(\theta_y(\mathrm{Z},\mathrm{CS}))}\,\middle|\,\mathrm{Z},\mathrm{S}\right]e^{-\theta_{y^*}^{wtr}Q(\mathrm{S})+\psi_Q(\theta_{y^*}^{wtr})}\,\middle|\,\mathrm{Z}\right]\right]$$

$$\approx\frac{1}{D}\sum_{d=1}^{D}1_{\{L(H)^{(d)}>y\}}e^{-\theta_y(\mathrm{Z}^{(d)},\mathrm{CS}^{(d)})L(H)^{(d)}+\psi_{L(H)|\mathrm{Z}^{(d)},\mathrm{CS}^{(d)}}(\theta_y(\mathrm{Z}^{(d)},\mathrm{CS}^{(d)}))}e^{-\theta_{y^*}^{wtr}Q(\mathrm{S}^{(d)})+\psi_Q(\theta_{y^*}^{wtr})}e^{-\sum_{c=1}^{C}\left(Z_c^{(d)}\mu_c-0.5\mu_c^2\right)}, \tag{71}$$

where $\theta_y(Z,\mathrm{CS})$ is given by (32) and – conditional on the realization of (Z,S) – the credit portfolio loss $L(H)$ is sampled according to the modified transition probabilities $h_{n,i,k}(\mathrm{Z},\mathrm{CS})$ (see (25)), $\theta_{y^*}^{wtr}$ is given by (68) and the vector of transformed market risk factors S is sampled according to $N(\mu(\theta_{y^*}^{wtr}),\Sigma(\theta_{y^*}^{wtr}))$ (see (62) and (63)) and the vector of systematic credit risk factors Z is sampled according to $N(\mu,\mathrm{I})$, where the C-dimensional mean vector μ is determined analogously to step 2 in section 3.2 as:

$$\underset{z_1,\dots,z_C\in\mathbb{R}}{\arg\max}\,E^{\tilde{P}_{\theta_{y^*}^{wtr}}}\left[E^{\tilde{P}_{\theta_y(\mathrm{z},\mathrm{CS})}}\left[1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{z},\mathrm{CS})L(H)+\psi_{L(H)|\mathrm{z},\mathrm{CS}}(\theta_y(\mathrm{z},\mathrm{CS}))}\,\middle|\,\mathrm{Z}=\mathrm{z},\mathrm{S}\right]e^{-\theta_{y^*}^{wtr}Q(\mathrm{S})+\psi_Q(\theta_{y^*}^{wtr})}\,\middle|\,\mathrm{Z}=\mathrm{z}\right]e^{-0.5\sum_{c=1}^{C}z_c^2}. \tag{72}$$

This optimization problem can be simplified by substituting the inner expectation by its upper boundary $e^{-\theta_y(\mathrm{z,CS})y+\psi_{L(H)|\mathrm{z,CS}}(\theta_y(\mathrm{z,CS}))}$, which yields for (72):

$$\underset{\mathrm{z}_1,\ldots,\mathrm{z}_C\in\mathbb{R}}{\arg\max}\, E^{\tilde{P}_{\theta_{y^*}^{wtr}}}\left[e^{-\theta_y(\mathrm{z,CS})y+\psi_{L(H)|\mathrm{z,CS}}(\theta_y(\mathrm{z,CS}))-\theta_{y^*}^{wtr}Q(\mathrm{S})+\psi_Q(\theta_{y^*}^{wtr})}\middle|\mathrm{Z}=\mathrm{z}\right]e^{-0.5\sum_{c=1}^{C}z_c^2}. \quad (73)$$

However, even this simplified optimization problem would be rather involved because there are usually many market risk factors which are relevant for the value of a portfolio and, hence, a multi-dimensional integral would have to be solved numerically many times in this optimization problem.

In order to circumvent this drawback an alternative might be to change the order in which the conditional expectations are computed:

$$\begin{aligned}
&P(L(H)>y)=E^{P}\left[P\left(L(H)>y\middle|\mathrm{Z},\mathrm{X}\right)\right]\\
&=E^{P}\left[E^{\tilde{P}_{\theta_y(\mathrm{Z,CS})}}\left[1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{Z,CS})L(H)+\psi_{L(H)|\mathrm{Z,CS}}(\theta_y(\mathrm{Z,CS}))}\middle|\mathrm{Z},\mathrm{S}\right]\right]\\
&=E^{\tilde{P}_{\theta_{y^*}^{wtr}}}\left[E^{P}\left[E^{\tilde{P}_{\theta_y(\mathrm{Z,CS})}}\left[1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{Z,CS})L(H)+\psi_{L(H)|\mathrm{Z,CS}}(\theta_y(\mathrm{Z,CS}))}\middle|\mathrm{Z},\mathrm{S}\right]\middle|\mathrm{S}\right]e^{-\theta_{y^*}^{wtr}Q(\mathrm{S})+\psi_Q(\theta_{y^*}^{wtr})}\right]\\
&=E^{\tilde{P}_{\theta_{y^*}^{wtr}}}\left[E^{P}\left[E^{\tilde{P}_{\theta_y(\mathrm{Z,CS})}}\left[1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{Z,CS})L(H)+\psi_{L(H)|\mathrm{Z,CS}}(\theta_y(\mathrm{Z,CS}))}e^{-\theta_{y^*}^{wtr}Q(\mathrm{S})+\psi_Q(\theta_{y^*}^{wtr})}\middle|\mathrm{Z},\mathrm{S}\right]\middle|\mathrm{S}\right]\right]\\
&=E^{\tilde{P}_{\theta_{y^*}^{wtr}}}\left[E^{P_{\mu(\mathrm{S})}}\left[E^{\tilde{P}_{\theta_y(\mathrm{Z,CS})}}\left[1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{Z,CS})L(H)+\psi_{L(H)|\mathrm{Z,CS}}(\theta_y(\mathrm{Z,CS}))}e^{-\theta_{y^*}^{wtr}Q(\mathrm{S})+\psi_Q(\theta_{y^*}^{wtr})}\middle|\mathrm{Z},\mathrm{S}\right]e^{-\sum_{c=1}^{C}\left(Z_c\mu_c(\mathrm{S})-0.5\mu_c(\mathrm{S})^2\right)}\middle|\mathrm{S}\right]\right],
\end{aligned}$$

(74)

where $P_{\mu(\mathrm{S})}$ is the multivariate normal distribution with mean vector $\mu(\mathrm{S})$ and the identity matrix I as covariance matrix. The C-dimensional IS mean vector $\mu=\mu(\mathrm{S})$ of the systematic credit risk factors Z, which now depends on the realization of the market risk factors S, can be determined as the solution of the following optimization problem:

$$\underset{\mathrm{z}_1,\ldots,\mathrm{z}_C\in\mathbb{R}}{\arg\max}\, E^{\tilde{P}_{\theta_y(\mathrm{z,CS})}}\left[1_{\{L(H)>y\}}e^{-\theta_y(\mathrm{z,CS})L(H)+\psi_{L(H)|\mathrm{z,CS}}(\theta_y(\mathrm{z,CS}))}e^{-\theta_{y^*}^{wtr}Q(\mathrm{S})+\psi_Q(\theta_{y^*}^{wtr})}\middle|\mathrm{Z}=\mathrm{z},\mathrm{S}\right]e^{-0.5\sum_{c=1}^{C}z_c^2}, \quad (75)$$

which is approximated by the solution of the following simplified optimization problem:

$$\underset{\mathrm{z}_1,\ldots,\mathrm{z}_C\in\mathbb{R}}{\arg\max}\, e^{-\theta_y(\mathrm{z,CS})y+\psi_{L(H)|\mathrm{z,CS}}(\theta_y(\mathrm{z,CS}))-\theta_{y^*}^{wtr}Q(\mathrm{S})+\psi_Q(\theta_{y^*}^{wtr})}e^{-0.5\sum_{c=1}^{C}z_c^2}$$

$$= \arg\max_{z_1,\dots,z_C\in\mathbb{R}} \left(-\theta_y(\mathrm{z},\mathrm{CS})y + \psi_{L(H)|\mathrm{z},\mathrm{CS}}(\theta_y(\mathrm{z},\mathrm{CS})) - \theta_{y*}^{wtr} Q(\mathrm{S}) + \psi_Q(\theta_{y*}^{wtr}) - 0.5\sum_{c=1}^{C} z_c^2 \right)$$

$$= \arg\max_{z_1,\dots,z_C\in\mathbb{R}} \left(-\theta_y(\mathrm{z},\mathrm{CS})y + \psi_{L(H)|\mathrm{z},\mathrm{CS}}(\theta_y(\mathrm{z},\mathrm{CS})) - 0.5\sum_{c=1}^{C} z_c^2 \right). \tag{76}$$

Unfortunately, this optimization problem has the serious disadvantage that it has to be solved for every scenario of the market risk factors S, which makes this approach computational expensive, too, and, hence, slow.

A third possibility to combine all three steps, which could be called a 'quick and dirty' approach, might be to assume for the determination of the IS means of the systematic credit risk factors Z that the market risk factors S equal there IS means $E^{\tilde{P}_{\theta_{y*}^{wtr}}}[S] = \mu(\theta_{y*}^{wtr})$. Then, the IS estimator would be given by:

$$P(L(H) > y)$$

$$\approx \frac{1}{D}\sum_{d=1}^{D} 1_{\{L(H)^{(d)}>y\}} e^{-\theta_y(\mathrm{Z}^{(d)},\mathrm{CS}^{(d)})L(H)^{(d)} + \psi_{L(H)|\mathrm{Z}^{(d)},\mathrm{CS}^{(d)}}(\theta_y(\mathrm{Z}^{(d)},\mathrm{CS}^{(d)}))} e^{-\theta_{y*}^{wtr}Q(\mathrm{S}^{(d)}) + \psi_Q(\theta_{y*}^{wtr})} e^{-\sum_{c=1}^{C}\left(Z_c^{(d)}\mu_c - 0.5\mu_c^2\right)}, \tag{77}$$

where again $\theta_y(\mathrm{Z},\mathrm{CS})$ is given by (32), conditional on the realization of (Z,S) the credit portfolio loss $L(H)$ is sampled according to the modified transition probabilities $h_{n,i,k}(\mathrm{Z},\mathrm{CS})$ (see (25)), θ_{y*}^{wtr} is given by (68) and the vector of transformed market risk factors S is sampled according to $N(\mu(\theta_{y*}^{wtr}),\Sigma(\theta_{y*}^{wtr}))$ (see (62) and (63)) and the vector of systematic credit risk factors Z is sampled according to $N(\mu,\mathrm{I})$, where the C-dimensional mean vector μ is now determined as:

$$\arg\max_{z_1,\dots,z_C\in\mathbb{R}} E^{\tilde{P}_{\theta_{y*}^{wtr}}}\left[E^{\tilde{P}_{\theta_y(\mathrm{z},\mathrm{CS})}}\left[1_{\{L(H)>y\}} e^{-\theta_y(\mathrm{z},\mathrm{CS})L(H) + \psi_{L(H)|\mathrm{z},\mathrm{CS}}(\theta_y(\mathrm{z},\mathrm{CS}))} \middle| \mathrm{Z}=\mathrm{z},\mathrm{S} \right] e^{-\theta_{y*}^{wtr}Q(\mathrm{S}) + \psi_Q(\theta_{y*}^{wtr})} \middle| \mathrm{Z}=\mathrm{z} \right] e^{-0.5\sum_{c=1}^{C} z_c^2}$$

$$\approx \arg\max_{z_1,\dots,z_C\in\mathbb{R}} E^{\tilde{P}_{\theta_{y*}^{wtr}}}\left[e^{-\theta_y(\mathrm{z},\mathrm{CS})y + \psi_{L(H)|\mathrm{z},\mathrm{CS}}(\theta_y(\mathrm{z},\mathrm{CS})) - \theta_{y*}^{wtr}Q(\mathrm{S}) + \psi_Q(\theta_{y*}^{wtr})} \middle| \mathrm{Z}=\mathrm{z} \right] e^{-0.5\sum_{c=1}^{C} z_c^2}$$

$$\approx \arg\max_{z_1,\dots,z_C\in\mathbb{R}} E^{\tilde{P}_{\theta_{y*}^{wtr}}}\left[e^{-\theta_y(\mathrm{z},\mathrm{C}\mu(\theta_{y*}^{wtr}))y + \psi_{L(H)|\mathrm{z},\mathrm{C}\mu(\theta_{y*}^{wtr})}(\theta_y(\mathrm{z},\mathrm{C}\mu(\theta_{y*}^{wtr}))) - \theta_{y*}^{wtr}Q(\mu(\theta_{y*}^{wtr})) + \psi_Q(\theta_{y*}^{wtr})} \middle| \mathrm{Z}=\mathrm{z} \right] e^{-0.5\sum_{c=1}^{C} z_c^2}$$

$$= \arg\max_{z_1,\dots,z_C\in\mathbb{R}} e^{-\theta_y(\mathrm{z},\mathrm{C}\mu(\theta_{y*}^{wtr}))y + \psi_{L(H)|\mathrm{z},\mathrm{C}\mu(\theta_{y*}^{wtr})}(\theta_y(\mathrm{z},\mathrm{C}\mu(\theta_{y*}^{wtr}))) - \theta_{y*}^{wtr}Q(\mu(\theta_{y*}^{wtr})) + \psi_Q(\theta_{y*}^{wtr}) - 0.5\sum_{c=1}^{C} z_c^2}$$

$$= \arg\max_{z_1,\dots,z_C\in\mathbb{R}} \left(-\theta_y(\mathrm{z},\mathrm{C}\mu(\theta_{y*}^{wtr}))y + \psi_{L(H)|\mathrm{z},\mathrm{C}\mu(\theta_{y*}^{wtr})}(\theta_y(\mathrm{z},\mathrm{C}\mu(\theta_{y*}^{wtr}))) - 0.5\sum_{c=1}^{C} z_c^2 \right). \tag{78}$$

The effectiveness of this approach is tested within the numerical example in the next section.

4 Numerical Example

Next, the effectiveness of the IS techniques derived in the previous section is analysed by means of numerical experiments. For this, the 'working example' of an integrated market and credit portfolio model as described in section 2.2 is employed.

4.1 Parameters

It is assumed that the credit portfolio consists of $N = 500$ defaultable zero coupon bonds, which are issued by N different obligors, but are otherwise identical. The face value is chosen to be $F = 1$ and the maturity date is $T = 3$, implying a remaining time to maturity of two years at the risk horizon. The simulations are done for the initial ratings $\eta_0 \in \{\text{Aa, Baa, B}\}$. As typical parameters for the *Vasicek* term structure model $\kappa = 0.4$ and $\sigma_r = 0.01$ are chosen. The mean level θ and the initial short rate $r(0)$ are set equal to 0.06. As market price of interest rate risk λ a value of 0.5 is taken.[27] The recovery rate is set equal to 53.80%, which is the mean of the recovery rate of senior unsecured bonds during 1970 to 1995 reported by Moody's.[28] The employed transition matrix is also from Moody's.[29] The values of the correlation parameter ρ_V of the asset returns are chosen as 0.1 and 0.4. The parameter $\rho_{r,V}$, which determines the correlation between the firms' asset returns and the term structure of interest rates, is set equal to $\rho_{r,V} = -0.05$. This value lies within the range of correlation parameters estimated in recent empirical studies of structural credit risk models.[30] The credit spreads are set equal to the credit spread means determined by *Kiesel*, *Perraudin* and *Taylor* (2003).[31]

4.2 Results

In the following the percentiles $\alpha_p(L(H))$ of the credit portfolio loss variable $L(H)$ as defined in (22) are computed for $p \in \{95\%, 99\%, 99.9\%, 99.98\%\}$. In each case, this is done with

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27 For example *Barnhill* and *Maxwell* (2002) estimate a short rate volatility of 0.007, whereas *Lehrbass* (1997) finds $\sigma_r = 0.029$ and *Huang* and *Huang* (2003) even work with $\sigma_r = 0.0468$. With regard to the mean reversion parameter and the market price of interest rate risk, *Lehrbass* finds $\kappa = 1.169$ and absolute values of 0.59, 0.808 and 1.232 for the parameter λ, whereas *Huang* and *Huang* choose $\kappa = 0.226$ and an absolute value of 0.248 for λ.

28 See *Moody's Investors Service* (1996). However, using Moody's estimate of the mean ignores the fact that the rating agency defines the recovery rate as a percentage of par and not as a percentage of a risk-free but otherwise identical zero coupon bond.

29 See *Moody's Investors Service* (2002, p. 31). The probabilities are average values of all corporates in the period 1970-2001. The category 'rating withdrawn' has been eliminated by distributing its probability mass among all other categories, corresponding to their individual weights.

30 See *Eom*, *Helwege* and *Huang* (2004, table 1, p. 505) and *Lyden* and *Saraniti* (2000, table 6, p. 38).

31 See *Kiesel*, *Perraudin* and *Taylor* (2003, table 1, p. 10, table 2, p. 18, appendix, p. 32).

and without an application of the IS technique.[32] Repeating these computations fifty times allows to calculate the standard error of the percentile estimators. Based on these standard errors, the ratio of the standard error of the percentile estimator without an application of IS and the respective standard error of the percentile estimator with IS is computed. These ratios allow to evaluate the effectiveness of the IS technique.

As an initial guess for the percentiles y which we are looking for the percentiles resulting from a crude pre-Monte Carlo simulation with a very low number of simulation runs (10,000) are used. These figures enter the IS estimators (51) and (77), respectively, for the excess probabilities $P(L(H) > y)$. The exact percentiles are calculated from these excess probabilities by a simple bisection method. For this, in each iterative step the arguments in the indicator functions in (51) and (77), respectively, are modified until the excess probability estimator equals one minus the confidence level with the desired precision. However, the y, on which the parameter $\theta_y(\mathrm{Z}, \mathrm{X})$ and the IS means of the systematic risk factors depend, are not altered during the iteration. Of course, the simulation of the credit portfolio loss variable also has to be done only once.

In the case that the third step as described in section 3.3 is also employed an additional pre simulation is carried out and the resulting guesses y^* for the percentiles are used for calculating the IS means and IS variances of the market risk factors (see (62), (63) and (68)). For this second pre-simulation the future ratings of the obligors are set equal to their current ratings so that portfolio losses are only due to changes in the market risk factors.

Table 1 shows the standard error ratios for the base case parameters described in section 4.1 when the two step-estimator (51) is employed. The most important observation is that the two step-IS technique is capable of reducing the standard error of the percentile estimators significantly and that, as expected, in general the reduction is larger the higher the confidence level of the percentile, even if there is no strict monotony. However, concerning the dependence of the standard error reduction on the credit quality or the asset return correlation no clear statements are possible because the results are rather mixed. Table 1 also shows the relative importance of IS for the systematic credit risk factor Z and the interest rate factor X_r: For high quality portfolios with a low stochastic dependence between the credit quality changes of the

32 Not using the IS technique corresponds to setting the IS means and the (conditional) twisting parameter (32) equal to zero and the IS standard deviations equal to one.

obligors IS for the interest rate factor X_r is more important, whereas for low credit qualities and/or high asset return correlations IS for the systematic credit risk factor Z is essential. These results are consistent with the findings of *Grundke* (2005b) who also observes that for high quality credit portfolios interest rate risk contributes most to the Value-at-Risk estimates, whereas for lower credit qualities, as expected, transition risk is more important.

In table 1 also the standard error ratios for estimators of the expected shortfall $E^P\left[L(H)\middle|L(H)>y\right]$ with $y=\alpha_p(L(H))$ ($p\in\{95\%,99\%,99.9\%,99.98\%\}$) can be seen. In general, the standard error reduction is strengthened for this risk measure although the suboptimal parameter $\theta_y(Z,X_r)$ and the suboptimal IS means of the systematic risk factors Z and X_r originally calculated for the estimation of excess probabilities are used (see section 3.2).

– insert table 1 about here –

Next, the influence of the model parameterization and the homogeneity assumptions on the variance reduction effect has been tested. As table 2 shows, the standard error ratios are rather robust with respect to changes in the amount of interest rate risk, the number of obligors, the correlation between the asset returns and the risk-free interest rates or the degree of homogeneity of the portfolio[33]. Some standard error ratios are higher than in the base case setting (see table 1), other are lower, but no systematic difference can be observed.

– insert table 2 about here –

Finally, the three step-IS estimator (77) has been implemented, where the IS mean of the systematic credit risk factor Z is calculated according to (78) and the computation of the IS mean and IS variance of the interest rate factor X_r is based on (62), (63) and (68). As the second derivative $\left.\frac{\partial^2 L^{wtr}(X_r,H)}{(\partial X_r)^2}\right|_{X_r=E^P[X_r]=0}$ is very small, the parameter λ_1 is nearly zero and, hence, the IS variance (63) remains, compared to the original probability measure, nearly unchanged one. Table 3 shows that for almost all considered credit qualities, asset return correlations and confidence levels the three step-IS technique yields worse standard error reductions than the two step-IS approach. Probably, one reason for this worse performance of the

33 In the case of an 'inhomogeneous portfolio composition', instead of assuming that all obligors have the same initial rating, the credit quality distribution of an 'average' credit portfolio, based on *Gordy* (2000, p. 132), is employed. Furthermore, inhomogeneous exposures of {0.1, 0.4, 0.9, 1.6, 2.5}, which are equally distributed in each rating grade, are assumed.

three step-IS technique is that the approach for computing the optimal parameter theta (see (32)) when calculating the IS mean of Z is not identical with the approach for computing this parameter during each of the simulation runs. In the former case, the interest rate factor is non-stochastic and set equal to its IS mean (see (78)), whereas in the latter case theta is chosen as a function of the realizations of the systematic risk factor Z and X_r, which can both vary in a free manner according their probability distribution. Hence, when determining the optimal IS mean of Z only a sub-optimal parameter theta (and not the optimal value characterized by 32)) is employed, from which the sub-optimality of the computed IS mean of Z follows.

– insert table 3 about here –

5 Conclusions

Standard credit portfolio models do not model market risk factors, such as risk-free interest rates or credit spreads, as stochastic variables. Various studies have documented that a severe underestimation of economic capital can be the consequence. However, integrating market risk factors into credit portfolio models increases the computational burden of computing credit portfolio risk measures, which makes the necessity of developing efficient computational methods for this type of credit portfolio model even more obvious.

In this paper, the application of various importance sampling techniques to an integrated market and credit portfolio model are presented and the effectiveness of these approaches for estimating percentiles of the credit portfolio loss variable, which are needed for Value-at-Risk calculations, as well as expected shortfalls is tested by means of numerical experiments. The main result is that importance sampling *can* reduce the dispersion of the estimators, but it is rather difficult to make statements about when the IS approach is especially effective. Only the fact that in general, as expected, the standard error reduction is larger the higher the confidence level of the percentile is obvious. Besides, it can be observed that the combination of importance sampling techniques originally developed for pure market risk portfolio models with techniques originally developed for pure default mode credit risk portfolio models is less effective than simpler two step-IS approaches.

Extensions of the study presented in this paper should for example check the robustness of the IS technique to altered portfolio compositions (for example other instrument types, e.g. options with counterparty risk) or the number of systematic risk factors. Furthermore, the effectiveness of the IS technique for calculating risk measures in the context of integrated market and credit portfolio models should be compared with that one of non-probabilistic approaches, such as e.g. Fourier based methods.

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TABLES

Table 1:
Standard error ratios for percentile and expected shortfall estimators with the two step-IS technique

confidence level	Aa $\rho_V = 0.1$	Aa $\rho_V = 0.4$	Baa $\rho_V = 0.1$	Baa $\rho_V = 0.4$	B $\rho_V = 0.1$	B $\rho_V = 0.4$
percentile estimators						
IS means for Z and X_r optimally chosen						
99.98%	38.4	5.4	22.5	27.6	50.0	31.1
99.9%	11.8	0.5	14.5	14.8	11.7	21.0
99%	4.5	2.2	6.2	1.5	5.7	7.4
95%	2.7	3.2	3.1	2.1	2.8	3.9
no IS for Z						
99.98%	30.3	1.9	3.8	0.8	1.7	1.2
99.9%	14.4	2.8	4.2	1.2	0.8	1.1
99%	5.3	1.3	2.2	1.2	1.1	1.2
95%	3.1	2.5	3.1	1.1	0.9	1.1
no IS for X_r						
99.98%	0.6	3.6	1.5	27.4	22.7	25.7
99.9%	0.6	0.4	1.3	12.4	10.2	12.2
99%	0.8	0.9	1.2	0.4	4.3	6.2
95%	1.0	1.2	1.3	0.6	2.0	3.7
expected shortfall estimators						
99.98%	216.8	1.1	25.0	75.6	138.1	300.8
99.9%	40.6	1.3	20.1	52.5	42.3	49.9
99%	10.2	1.0	6.3	10.3	11.2	8.7
95%	5.6	1.9	4.9	2.2	5.8	4.6

Notes: Standard error ratios, defined as the standard error of the percentile estimator of the loss variable $L(H)$ (expected shortfall estimator $E^P\left[L(H)\middle|L(H)>\alpha_p\left(L(H)\right)\right]$) without an application of IS divided by the standard error of the respective estimator with IS, are shown. The computations of the standard errors are based on fifty repetitions of the simulations (with and without IS), where each simulation consists of 10,000 simulation runs. *Notation*: ρ_V: asset return correlation. *Parameters*: $N = 500$, $F = 1$, $T = 3$, $H = 1$, $\rho_{r,V} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

Table 2:
Standard error ratios for percentile estimators with the two step-IS technique for varying parameterizations and portfolio compositions

confidence level	Aa		Baa		B	
	$\rho_V = 0.1$	$\rho_V = 0.4$	$\rho_V = 0.1$	$\rho_V = 0.4$	$\rho_V = 0.1$	$\rho_V = 0.4$
$\sigma_r = 0$ (no interest rate risk)						
99.98%	36.5	39.7	40.5	38.3	43.4	19.5
99.9%	18.2	14.3	14.3	15.6	15.9	14.4
99%	7.4	6.1	5.7	5.8	6.6	4.6
95%	3.4	3.0	2.9	3.7	3.8	2.7
$\rho_{r,V} = -0.25$ (increased negative correlation between asset returns and interest rates)						
99.98%	34.2	31.3	35.1	50.4	34.1	27.4
99.9%	18.9	7.3	12.8	15.7	18.8	19.6
99%	7.2	2.4	6.8	5.8	5.5	5.4
95%	4.1	3.0	2.8	2.7	2.8	3.1
$N = 50$ (reduced number of obligors)						
99.98%	40.8	2.0	26.6	59.6	33.6	37.4
99.9%	13.5	2.6	12.3	21.4	14.2	21.5
99%	6.9	3.0	5.7	4.0	6.8	7.5
95%	3.6	3.0	3.2	1.8	3.4	3.5
inhomogeneous portfolio composition						
99.98%	23.2	20.8				
99.9%	25.4	12.7				
99%	7.0	4.5				
95%	2.2	5.5				

Notes: Standard error ratios, defined as the standard error of the respective percentile estimator of the loss variable $L(H)$ without an application of IS divided by the respective standard error of the percentile estimator with IS, are shown. The computations of the standard errors are based on fifty repetitions of the simulations (with and without IS), where each simulation consists of 10,000 simulation runs. In the case of an 'inhomogeneous portfolio composition', instead of assuming that all obligors have the same initial rating, the credit quality distribution of an 'average' credit portfolio, based on *Gordy* (2000, p. 132), is employed. Furthermore, inhomogeneous exposures of {0.1, 0.4, 0.9, 1.6, 2.5}, which are equally distributed in each rating grade, are assumed. *Notation*: ρ_V : asset return correlation. *Parameters (as far as not otherwise indicated)*: $N = 500$, $F = 1$, $T = 3$, $H = 1$, $\rho_{r,V} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.

Table 3:
Standard error ratios for percentile estimators with the three step-IS technique

confidence level	Aa		Baa		B	
	$\rho_V = 0.1$	$\rho_V = 0.4$	$\rho_V = 0.1$	$\rho_V = 0.4$	$\rho_V = 0.1$	$\rho_V = 0.4$
99.98%	22.0	1.1	9.2	0.8	3.6	1.8
99.9%	4.5	1.8	5.5	0.8	1.6	0.9
99%	3.8	2.7	6.0	1.4	1.5	0.8
95%	3.0	2.8	3.4	2.0	1.4	1.0

Notes: Standard error ratios, defined as the standard error of the respective percentile estimator of the loss variable $L(H)$ without an application of IS divided by the respective standard error of the percentile estimator with IS, are shown. The computations of the standard errors are based on fifty repetitions of the simulations (with and without IS), where each simulation consists of 10,000 simulation runs. The IS means of the systematic credit risk factor Z are calculated according to (78), whereas the computations of the IS means and variances of the interest rate factor X_r are based on (62), (63) and (68). *Notation*: ρ_V: asset return correlation. *Parameters*: $N = 500$, $F = 1$, $T = 3$, $H = 1$, $\rho_{r,V} = -0.05$, $\delta = 0.538$, $\kappa = 0.4$, $\theta = 0.06$, $\sigma_r = 0.01$, $\lambda = 0.5$, $r(0) = 0.06$.