Anatomy of Estimation Approaches for the Statistical Uncertainty of a Long-Run Probability of Default

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Abstract

In this article we present and compare three common types of quantification approaches for the statistical uncertainty in the estimation of obligor default probabilities. Our results show that, in case of low correlation among obligors, the approaches yield rather similar results. In the case of correlated obligors, the uncertainty estimation itself is uncertain in the sense that either a model-dependent approach with a distribution assumption and an assumed (e.g. regulatory) or estimated strength of the correlation needs to be used or a very long time series must be available to produce reliable results based only on empirical data. In a low-default portfolio, or typical populations observed in practice with portfolio default rates of $\sim \mathcal{O}(1\%)$, the time series actually needs to be unrealistically long. Bootstrapping approaches do not improve on this situation.

An obvious solution to the issue of unreliability in situations with data scarcity and significant correlation is the use of high confidence levels, which comes with the unwanted consequence of large conservative add-ons to the PD estimates. The use of a distribution-based approach with suitable - and sufficiently conservative - assumptions on the distribution of defaults is, from our point of view, an attractive alternative to more empirical approaches in these situations. In a regulated environment, where the historical time window used for the estimation is not arbitrary but fixed by rules or an algorithm, a distribution-based approach with the use of a conditional variance constitutes an elegant solution.

Keywords: Probability of Default (PD); Statistical Uncertainty of PD; Estimation Error; Margin of Conservatism; IRB Approach; MoC C.

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1 Introduction

In this article, we compare different approaches to determine the statistical uncertainty of obligor probabilities of default (PDs) determined from time series of historical default rates. Our quantitative results are focussed on **size** and **accuracy** of confidence intervals (CI) around the observed long-run average default rate (LRADR) as the estimator for the (through-the-cycle) PD. In particular, we are interested in the impact of the length of the available time series on size and accuracy of CIs, given certain levels of default probability and correlation.

In the internal ratings-based (IRB) approach to calculate capital requirements for credit risk, banks are required to add a Margin of Conservatism (MoC) to their PD estimates to account for statistical and other uncertainties. In the nomenclature of the Guidelines [EBA, 2017] issued by the European Banking Authority (EBA), the statistical uncertainty or "general estimation error" requires a MoC of type C.

The quantification of the statistical uncertainty of PDs was being discussed in the literature during the time the IRB approach was introduced in 2006 (see e.g. [Schuermann and Hanson, 2004, Cantor et al., 2007, Roesch, 2004] and became a controversial topic of discussion again among practitioners in European banks after the publication of the EBA Guidelines [EBA, 2017] and increased supervisory scrutiny of quantification approaches. These discussions have led to new research output. Recent papers investigating different aspects of this topic are [Casellina et al., 2021], [Scherer et al., 2023], [Wosnitza, 2023], [Wosnitza, 2025].

One of the authors of this article has discussed the quantification of MoC C in the framework of the asymptotic single-risk factor (ASRF) model, a.k.a. Vasicek model [Vašiček, 2002], which is the basis of the risk-weight formulas of the IRB approach, in a previous publication [Scherer et al., 2023]. This article can be considered a follow-up to this previous publication and is partly based on results published therein. We use the same notation, with $R_L = \frac{1}{T}\sum_{t=1}^{T} R_D(t)$ being the LRADR as a random quantity and

$$r_L = \frac{1}{T}\sum_{t=1}^{T} r_D(t) \tag{1}$$

defining its observed value, whereas the $r_D(t)$ are the observed values of the random default rates $R_D(t)$ belonging to reference dates $t = 1, .., T$ (e.g. annual reference dates) with the time between two reference dates being the period during which the defaults are observed. Hence, T and the number of non-defaulted obligors $N(t)$ at the reference dates determine the sample size such that for $N(t), T \to \infty$ the statistical uncertainty should converge to zero.

The outline of this paper is as follows: In section 2, we give a brief overview of the three common types of approaches to quantify the statistical long-run PD uncertainty. In sections 3, 4 and 5, we study each of the three approaches in more detail, highlighting particular points of interest. Section 6 contains the main results of this paper, namely a comparison of the three approaches in terms of strengths and weaknesses and their quantitative outcomes in different scenarios. In section 7, we summarize our conclusions.

In Appendix D, we discuss how the novel approach published recently in [Wosnitza, 2023] fits into the overall picture.

2 Overview of approaches

In a recent client survey among 38 European IRB banks (available on request from the authors of this paper), Deloitte found three common types of approaches in the determination of MoC C:

- **Type I: Approaches based on a distribution assumption for the LRADR.** From an assumed distribution of the LRADR, one can derive moment estimators (e.g. the standard deviation) or CIs as measures of uncertainty. This is particularly simple when defaults are assumed to be *uncorrelated*, i.e. the default rate $R_D(t)$ follows a *binomial distribution* with the uniform obligor default probability p_{binom} and the number of obligors $N(t)$. We note that, in a portfolio with heterogeneous obligors, the estimation is usually performed on the level of rating grades or segments of obligors with similar risk characteristics such that the assumption of a uniform default probability is a reasonable approximation.

  More realistic than uncorrelated defaults is the assumption that defaults are *correlated* via systematic (external) risk factors linking all obligors to the economic cycle, i.e. the default rates $R_D(t)$ at different points in time have different conditional expectation values, depending on the realization of the systematic risk factors at t. A simple and elegant model fulfilling this assumption is the ASRF model, which links all obligors to one single systemic risk factor $\epsilon(t)$, albeit without allowing for autocorrelation in the time series of $\epsilon(t)$. We will use the ASRF model as an example of Type I approaches later in our analysis.

- **Type II: Approaches based on the empirical variance of default rates over time.** Assuming statistical independence of the default rates $R_D(t)$, one can easily relate the variance of R_L as a measure of its statistical uncertainty to the variance of $R_D(t)$. For the latter, the observed sample variance of historical $r_D(t)$ an be used as an estimator. The sample variance is expected to increase with increasing default correlation, so this type of approach implicitly reflects observed correlation. However, the accuracy and certainty of the results of approach obviously depend on the length of the available time series. We will investigate on this dependence and required length of time series in our analysis.

- **Type III: Approaches based on bootstrapping from observed default rates.** Starting from the time series of observed default rates $r_D(t)$, one can use bootstraping techniques to generate more LRADR values. The width of the distribution of these values will then be considered as a measure for the uncertainty of the LRADR. Similarly to the Type II approaches, the accuracy and certainty of the result depends on the length of the available time series and will be investigated later on.

In the following sections, we study all three types of approaches in more detail and compare them to one another. No preference for a particular type of approach has been published so far by the European supervisory authorities. While banks and other parties explicitly asked for more detailed guidance on MoC C quantification and the treatment of default correlations during the 2024 consultation process on the ECB guide to internal models (see e.g. [Wosnitza, 2024]), ECB did not provide additional details in their feedback statement, see comment 5 in section 5.8 of [ECB, 2024].

3 Approaches based on a distribution assumption (Type I)

3.1 Theory: Variance of the long-run average default rate

In the simplest case of a Type I approach, defaults are considered to be *uncorrelated*, i.e. the default rate $R_D(t)$ follows a binomial distribution with parameters p_{binom} as uniform default probability and $N(t)$. The variance of R_L is then

$$\mathrm{Var}(R_L) = \frac{1}{T^2}\sum_{t=1}^{T}\frac{p_{\text{binom}}(1-p_{\text{binom}})}{N(t)} \tag{2}$$

In the ASRF model with *correlated defaults*, a default rate $R_D(t)$ conditional on the realization $\epsilon(t)=\epsilon_t$ of the systematic factor at time t follows the distribution derived in [Vašiček, 2002], with the conditional expectation value

$$\mathrm{E}_c(R_D(t)) = \Phi\left(\frac{\Phi^{-1}(p)-\sqrt{\rho}\epsilon_t}{\sqrt{1-\rho}}\right). \tag{3}$$

where $\Phi(x)$ is the cumulative standard normal distribution and ρ the asset correlation of the obligors. Assuming correlated defaults, two possible perspectives on the variance of the LRADR exist. On the one hand, the *conditional variance*, taking a time window $t=1,\ldots,T$ and certain realizations ϵ_t as given, reads

$$\mathrm{Var}_c(R_L) = \frac{1}{T^2}\sum_{t=1}^{T}\frac{\mathrm{E}_c(R_D(t))\cdot(1-\mathrm{E}_c(R_D(t)))}{N(t)} \tag{4}$$

with $\mathrm{E}_c(R_D(t))$ given above and is similar to the variance derived from the binomial distribution because, conditional on ϵ_t, defaults of different obligors at time t are uncorrelated with one another in the ASRF model. On the other hand, the *unconditional variance* is (see derivation in [Scherer et al., 2023])

$$\mathrm{Var}(R_L) = \frac{1}{T^2}\sum_{t=1}^{T}\frac{p-\Phi_2\left(\Phi^{-1}(p),\Phi^{-1}(p);\rho\right)}{N(t)} + \frac{\Phi_2\left(\Phi^{-1}(p),\Phi^{-1}(p);\rho\right)-p^2}{T} \tag{5}$$

with Φ_2 being the bivariate cumulative standard normal distribution. We note that the formula given here is slightly more general than eq. (22) of [Scherer et al., 2023] because we have allowed for a variation of $N(t)$ over time.

The interpretation of eq. (5) is as follows: The first term reflects uncertainty related to the finite number of borrowers $N(t)$ and is the expectation value of the conditional expectation values of eq. (4), taking the standard normal distribution of $\epsilon(t)$ into account. The second term reflects additional uncertainty resulting from the correlation of borrowers and the randomness of $\epsilon(t)$ and does not depend on $N(t)$. The unconditional variance must be used if the time window $t=1,\ldots,T$ is considered to be a random selection among different possible windows, each one leading to a different observation r_L.

For the determination of MoC C in a regulatory context, the preference for one of these two perspectives is unclear. Recent ECB comments in [ECB, 2024] have not increased clarity, since they refer to "the statistical uncertainty of each one-year default rate and the length of the time series" as the primary sources of uncertainty, which holds true for both eq. (4) and eq. (5). No explicit requirement for the consideration of default correlations exists so far, although "the importance of analysing other relevant assumptions specific to the institution's chosen method" is stressed, which could entail an analysis of default correlation.

3.2 Practice: Margin of conservatism

All of eqs. (2), (4) and (5) are model-based and have model parameters as inputs. In practice, in order to estimate a Margin of Conservatism, these parameters need to be replaced by the best available estimators:

$$p_{\text{binom}} \to r_L, \qquad \mathrm{E}_c(R_D(t)) \to r_D(t), \qquad p \to r_L, \qquad \rho \to \hat{\rho} \tag{6}$$

with a suitable estimator $\hat{\rho}$ for the asset correlation. Various methods have been discussed in the literature to estimate ρ from default data but each of them comes with its own uncertainty that one needs to be aware of, so we advocate for the use of a conservative estimate of ρ in practical applications of eq. (5) and approaches relying on distributions assumptions different from the ASRF model.

As a MoC for statistical uncertainty, practitioners frequently use a multiple of the standard deviation of the LRADR by taking the square root of one of eqs. (2), (4) or (5) or a CI derived from the standard deviation (e.g. a "Wald interval"). Since the LRADR is an arithmetic average of default rates, the central limit theorem can be used for this derivation if the time series is sufficiently long. Relying on eq. (5), the classical two-sided Wald interval at confidence level $1-\alpha$ with upper bound $R_L^{u,\alpha}$ and lower bound $R_L^{l,\alpha}$ reads

$$R_L^{u/l,\alpha} = r_L \pm \Phi^{-1}\left(1-\frac{\alpha}{2}\right) \cdot \sqrt{\mathrm{Var}(R_L)}\Big|_{p=r_L}. \tag{7}$$

To be more conservative, in case of a short time series the Student t distribution with $T-1$ degrees of freedom should be used, see eq. (24) of [Roesch, 2004], so the same CI reads

$$R_L^{u/l,\alpha} = r_L \pm t_{T-1}^{-1}\left(1-\frac{\alpha}{2}\right) \cdot \sqrt{\mathrm{Var}(R_L)}\Big|_{p=r_L}. \tag{8}$$

For our analyses in section 6, we define a MoC C in relative terms, i.e. as a percentage of the LRADR, representative of Type I approaches as

$$\mathrm{MoC}_{\text{Type1}} = \frac{R_L^{u,\alpha=10\%} - r_L}{r_L} = \frac{t_{T-1}^{-1}(0.95) \cdot \sqrt{\mathrm{Var}(R_L)}\Big|_{p=r_L}}{r_L} \tag{9}$$

with $\mathrm{Var}(R_L)$ from eq. (5).

A potential issue is that, for small values of T, the LRADR distribution is right-skewed for $\rho > 0$, i.e. most of the time we will observe a LRADR r_L below the theoretical value p but values much higher than p are possible in the tail of the distribution. Symmetric CIs can therefore have a lower-than-expected coverage ratio. As an illustration of this effect, we display histograms of 5000 ASRF model simulations of r_L for $\rho = 0.2$ and $p = 1\%$ for $T = 10$, 25 and 50 in figure 1. The skewness of the distribution is obvious, in particular for the case $T = 10$, where values as high as $r_L > 3\%$ occur. In all three cases, the mode is below 1%. If default rates are measured on an annual basis, unrealistically long time series would hence be required to expect a symmetric LRADR distribution for strongly correlated defaults.

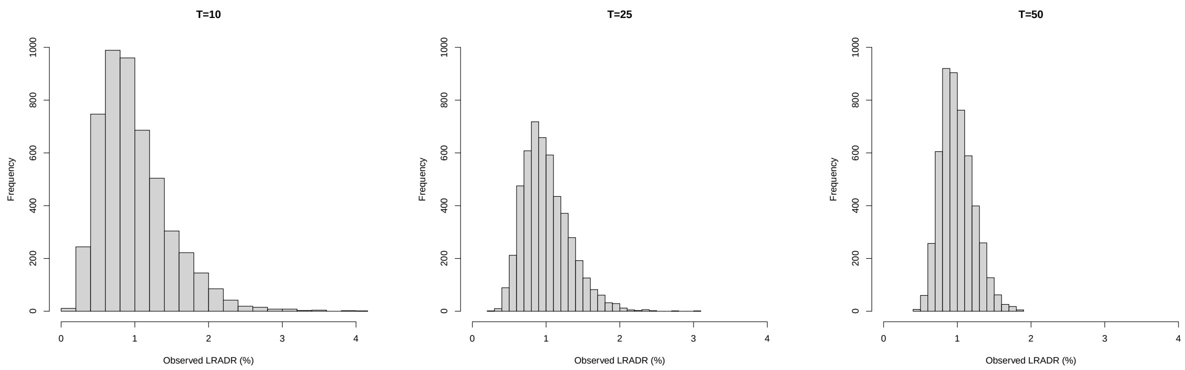

Figure 1: Histogram of the observed LRADR in 5000 ASRF model simulations with $N = 5000$, $p = 1\%$, $\rho = 0.2$ and $T = 10$ (left), $T = 25$ (center) and $T = 50$ (right)

We will investigate the impact of this behaviour on the coverage ratio of CIs in more detail in section 6.

4 Approaches based on the empirical variance of default rates (Type II)

Assuming statistical independence[1] of the default rates $R_D(t)$ and assuming they are random draws from one single unconditional distribution, one can easily relate the variance of R_L as a measure of its statistical uncertainty to the variance of $R_D(t)$, which is constant in this case:

$$\mathrm{Var}(R_L) = \frac{1}{T^2}\sum_{t=1}^{T}\mathrm{Var}(R_D(t)) = \frac{1}{T}\mathrm{Var}_{R_D} \tag{10}$$

Since the sample variance is an unbiased estimator for the population variance, we can use the sample variance of the default rates $r_D(t)$ observed at different points in time ("empirical variance"), $s^2_{r_D}$, as an estimator for Var_{R_D}, hence the estimator for $\mathrm{Var}(R_L)$ is

$$\widehat{\mathrm{Var}(R_L)} = \frac{1}{T}s^2_{r_D} \tag{11}$$

This can be used to estimate the uncertainty of the PD estimate, e.g. in terms of a CI, and only requires the observed default rates as input, but no specific assumption about default correlation in contrast to the distribution-based approaches of section 3. A simple Wald interval comparable to the one defined in eq. (7) is

$$R_L^{u/l,\alpha} = r_L \pm \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\cdot\sqrt{\frac{1}{T}s^2_{r_D}} \tag{12}$$

or, in a more conservative variant analogous to eq. (8),

$$R_L^{u/l,\alpha} = r_L \pm t^{-1}_{T-1}\left(1-\frac{\alpha}{2}\right)\cdot\sqrt{\frac{1}{T}s^2_{r_D}}\ . \tag{13}$$

A caveat is that such a CI can only be reliable if default rates are observed over a sufficiently long period. Since default correlation is expected to not only increase the variance of default rates but also make it more volatile, the period should be even longer in case defaults are expected to be correlated. This can easily be seen by inspecting the distribution of the square root of eq. (11), which is almost symmetric for $\rho = 0$ but strongly right-skewed for large values of ρ. For illustration, we display simulated distributions of this quantity in figure 2, using 5000 simulations of the ASRF model for each parameter combination of $p = 1\%$, $T = 10$ and four different values of ρ. Intuitively, this behaviour is understandable since the impact of macroeconomic outliers on the LRADR variance grows with increasing ρ.

[1] As a minimum, this means non-overlapping observation periods for defaults. Recently, a method to derive the variance in case of overlapping observation periods from the non-overlapping variance was derived in [Wosnitza, 2025].

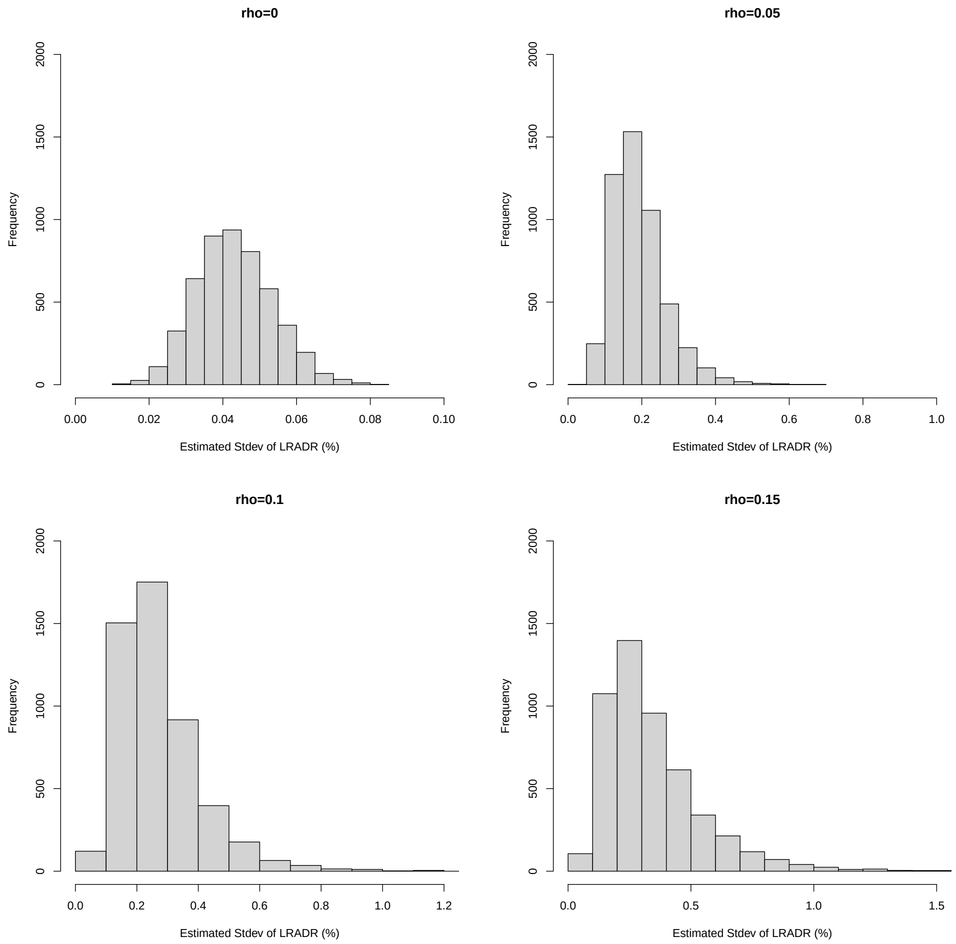

Figure 2: Histogram of the square root of eq. 11 in 5000 simulations with $N = 5000$, $p = 1\%$, $T = 10$ and $\rho = 0$ (top left), $\rho = 0.05$ (top right), $\rho = 0.10$ (bottom left) and $\rho = 0.15$ (bottom right)

We will investigate the impact of this behaviour on the coverage ratio of CIs in more detail in section 6 to see what the minimum length of a time series should be to obtain reliable results and if bootstrapping (see next section) can help to improve the situation. For the purpose of these analyses, we define a MoC C in relative terms, i.e. as a percentage of the LRADR, for the Type II approach as

$$\mathrm{MoC}_{\mathrm{Type2}} = \frac{R_L^{u,\alpha=10\%} - r_L}{r_L} = \frac{t_{T-1}^{-1}(0.95) \cdot \sqrt{\frac{1}{T} s_{r_D}^2}}{r_L}. \tag{14}$$

with $R_L^{u,\alpha=10\%}$ calculated via eq. (13).

5 Approaches based on bootstrapping (Type III)

Similarly to the Type II approaches, bootstrapping only requires the observed default rates as input, without any assumptions on the default correlation or the underlying default mechanism. From the observed time series of default rates $r_D(t)$, one generates a sufficiently large number of new time series via random sampling with replacement, such that multiple values of r_L can be calculated. From this simulated distribution of r_L, one can calculate confidence intervals in terms of percentiles of the distribution. For our analyses, we will use such a CI lying between the 5th percentile $P_5(r_L)$ and the 95th percentile $P_{95}(r_L)$ of the distribution such that

$$\mathrm{MoC}_{\mathrm{Type3}} = \frac{P_{95}(r_L) - r_L}{r_L}. \tag{15}$$

In order to produce stable results in our ASRF model simulation analyses of section 6, we strived to find a reasonable minimum number of bootstraps per simulation run. To this end, we tried different numbers of bootstraps between 50 and 1500 for different numbers of simulation runs between 50 and 15000 while also varying the model parameters p, ρ and T. For every combination of model parameter values, we looked for the point where the coverage ratio of the simulated 90% CI stabilizes. Depending on the model parameter values, the stabilization happens at a value more or less close to the nominal value of 90%. Some sample results are depicted in figure 3.

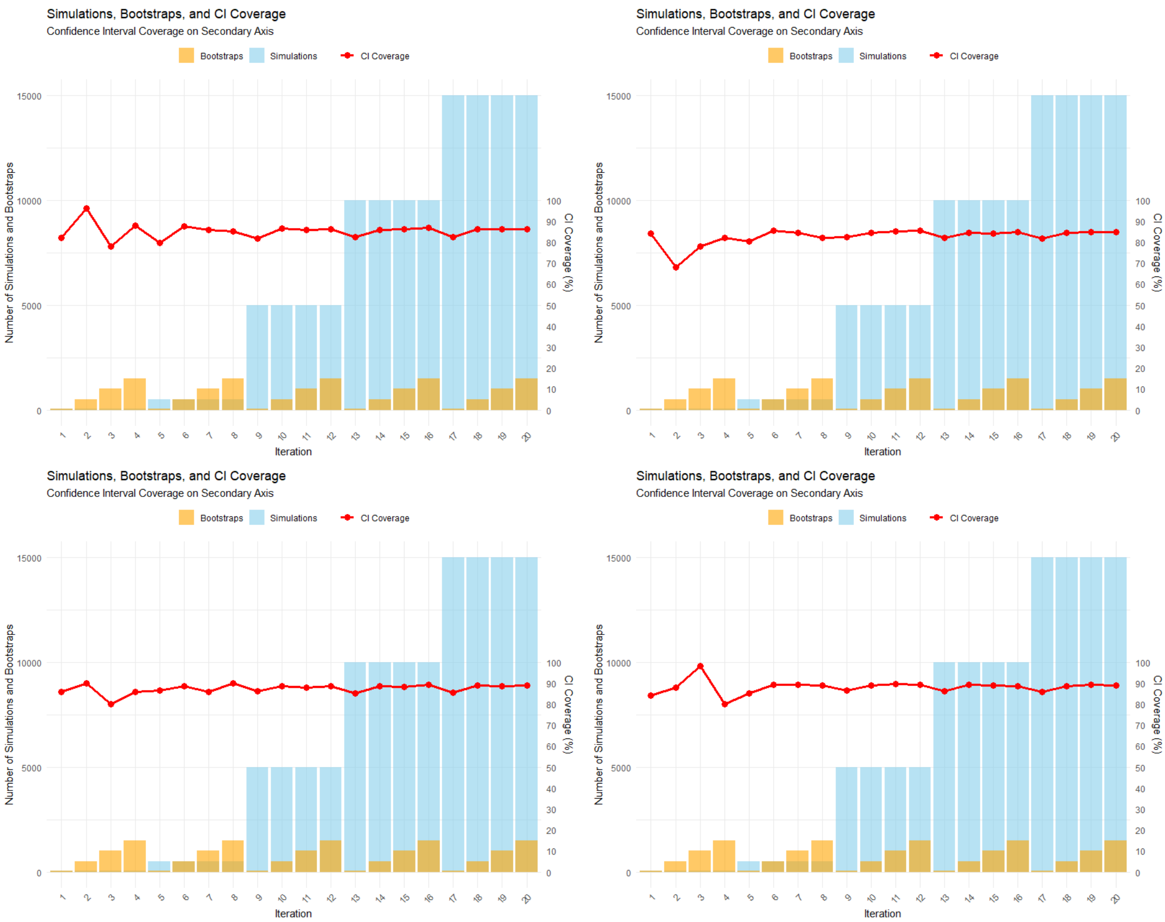

Figure 3: Number of simulations and number of bootstraps (left y-axis) and coverage ratio of the 90% CI (right y-axis) for $p = 0.1\%$, $\rho = 0$ and $T = 10$ (top left), $p = 1\%$, $\rho = 0.1$ and $T = 25$ (top right), $p = 10\%$, $\rho = 0.05$ and $T = 50$ (bottom left), $p = 10\%$, $\rho = 0.2$ and $T = 100$ (bottom right)

We find that 1000 bootstraps per simulation run in 5000 overall simulations is sufficient to produce stable results for all model parameter combinations. This also allows us to compare the Type III results to the results obtained from Type I and Type II approaches in section 6 using 5000 ASRF model simulations and investigate to what extent bootstrapping helps to improve the uncertainty estimation results.

6 Comparison of approaches

6.1 Overview

Throughout section 6, we compare the three types of approaches discussed in this paper in terms of size and coverage of confidence intervals around the LRADR. To this end, we simulate random defaults in a portfolio of $N = 5000$ obligors by means of the ASRF model. We thereby vary the asset correlation ρ between 0 (uncorrelated defaults) and 0.2 (correlated defaults). We also vary the model parameters p to obtain insights into the estimation uncertainty for both high-default and low-default populations. Additionally, varying the length T of the time series (in our case the number of non-overlapping observation periods) of simulated default rates allows us to learn how the three types of approaches behave in different situations of data availability. In detail, we use the following varying input parameters:

$$\begin{array}{lll} \rho & \epsilon & \{0,\ 0.05,\ 0.1,\ 0.15,\ 0.2\} \\ p & \epsilon & \{0.1\%,\ 1\%,\ 10\%\} \\ T & \epsilon & \{10,\ 25,\ 50,\ 75,\ 100,\ 150,\ 200\}. \end{array}$$

Since we tend to interpret T as a number of non-overlapping 12-month observation periods as frequently used in practice in banks, we omit values $T > 75$ as unrealistic in graphical representations of our results.

As discussed in the previous section, we fix the number of simulations at $n_{\text{sim}} = 5000$ and the number of bootstraps for the Type III approach at $n_{\text{boot}} = 1000$. Every simulation run corresponds to an observed default rate time series in reality from which a bank would calculate a MoC. Hence, for every simulation run, we calculate a Type I MoC by means of equation (9) and a Type II MoC through eq. (14). Moreover, after bootstrapping, we can calculate a Type III MoC per simulation run with eq. (15).

For the comparison of results, it is important to note the following key considerations:

1. Type I approaches depend on some assumptions. In our case, we treat the asset correlation ρ as a known input parameter of the ASRF model. In practice, the estimation of this parameter would generate additional uncertainty.

2. According to our definitions, Type I and II CIs are symmetric around r_L. In low-default populations, their lower bound can be negative, which is clearly misleading. By contrast, Type III CIs can be highly asymmetric in these situations but their lower bound can, by definition, not be negative.

In the following subsections, we first give an overview of the sizes of median MoC values and of the coverage ratios of the confidence intervals from which the MoCs are derived before we present the results in more detail in sections 6.2 to 6.4.

6.1.1 Overview of median MoC C levels

For each of the three MoC types, we show the median value resulting from the 5000 simulations for different simulation input configurations in figure 4.

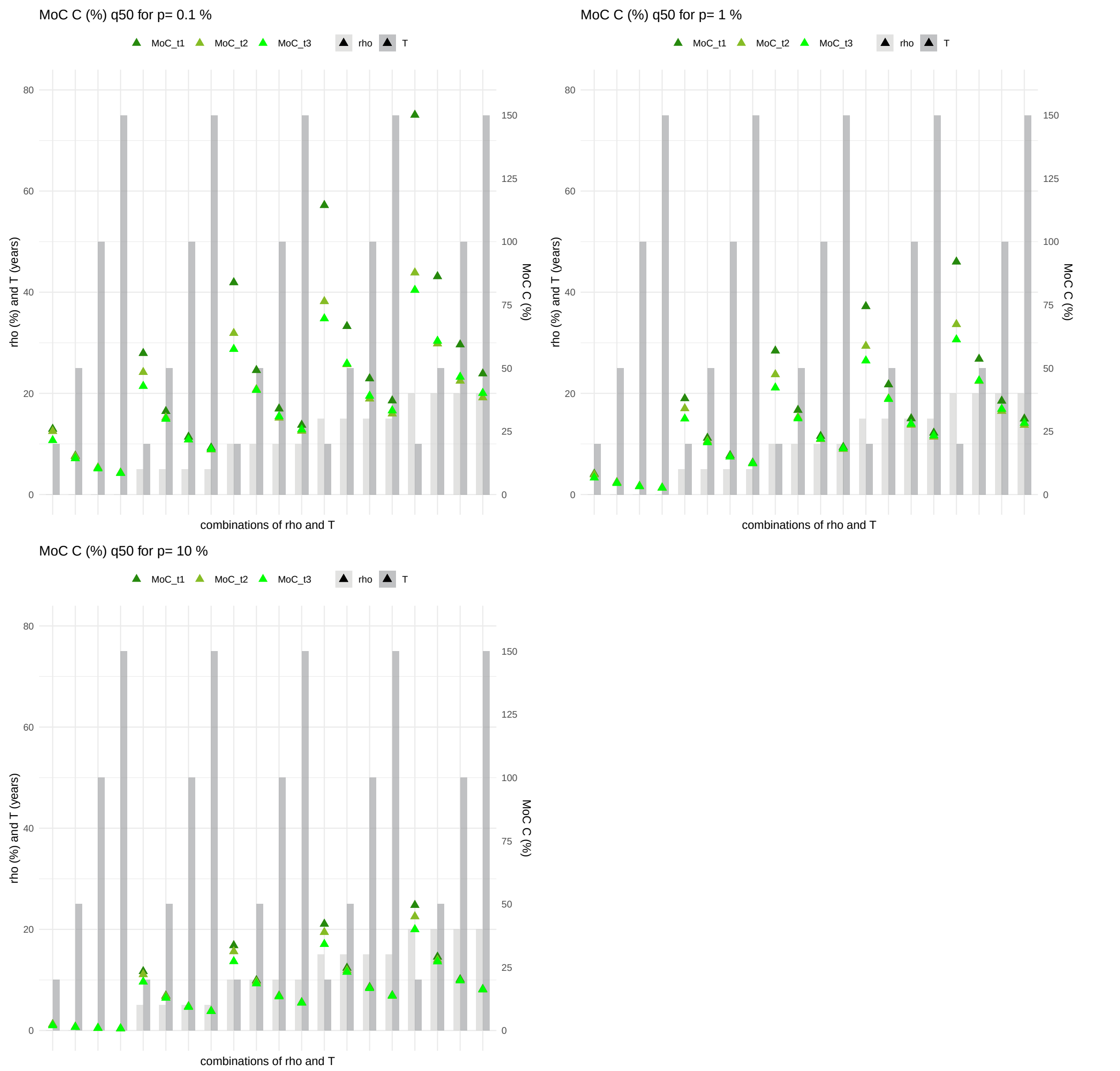

Figure 4: Median value of simulated (relative) MoC C (right y-axis) for Type I, II and III for different values of ρ and T (left y-axis). Top row: $p = 0.1\%$ (left), $p = 1\%$ (right). Bottom: $p = 10\%$.

Our key findings are:

- As expected, in all approaches the relative MoCs strongly increase with increasing ρ and decreasing T. They are also significantly larger for low-default populations (small p) than for high-default populations.
- The differences between the approaches are most pronounced in low-default populations, see the plot for $p = 0.1\%$. However, in case of uncorrelated defaults (ρ=0), they small for all values of p.
- Type I consistently produces the highest median values especialy for low default populations and short time series, see plot for $p = 0.1\%$ and $p = 1\%$. We note that differences between Type II and III are mostly due to our conservative use of the t distribution instead of the normal distribution in the CI derivation following eq. (14).

6.1.2 Overview of coverage ratios of the confidence intervals

The use of 5000 simulations allows us to study the actual coverage ratio of the two-sided 90% CIs which we have used to define the MoCs. We define the coverage ratio to be the number of times the "true" LRADR, which in our case is the ASRF parameter p, is contained in the CI, divided by n_{sim}. Ideally, one would expect a 90% coverage ratio per definition. However, figure 5 shows that this is not always the case.

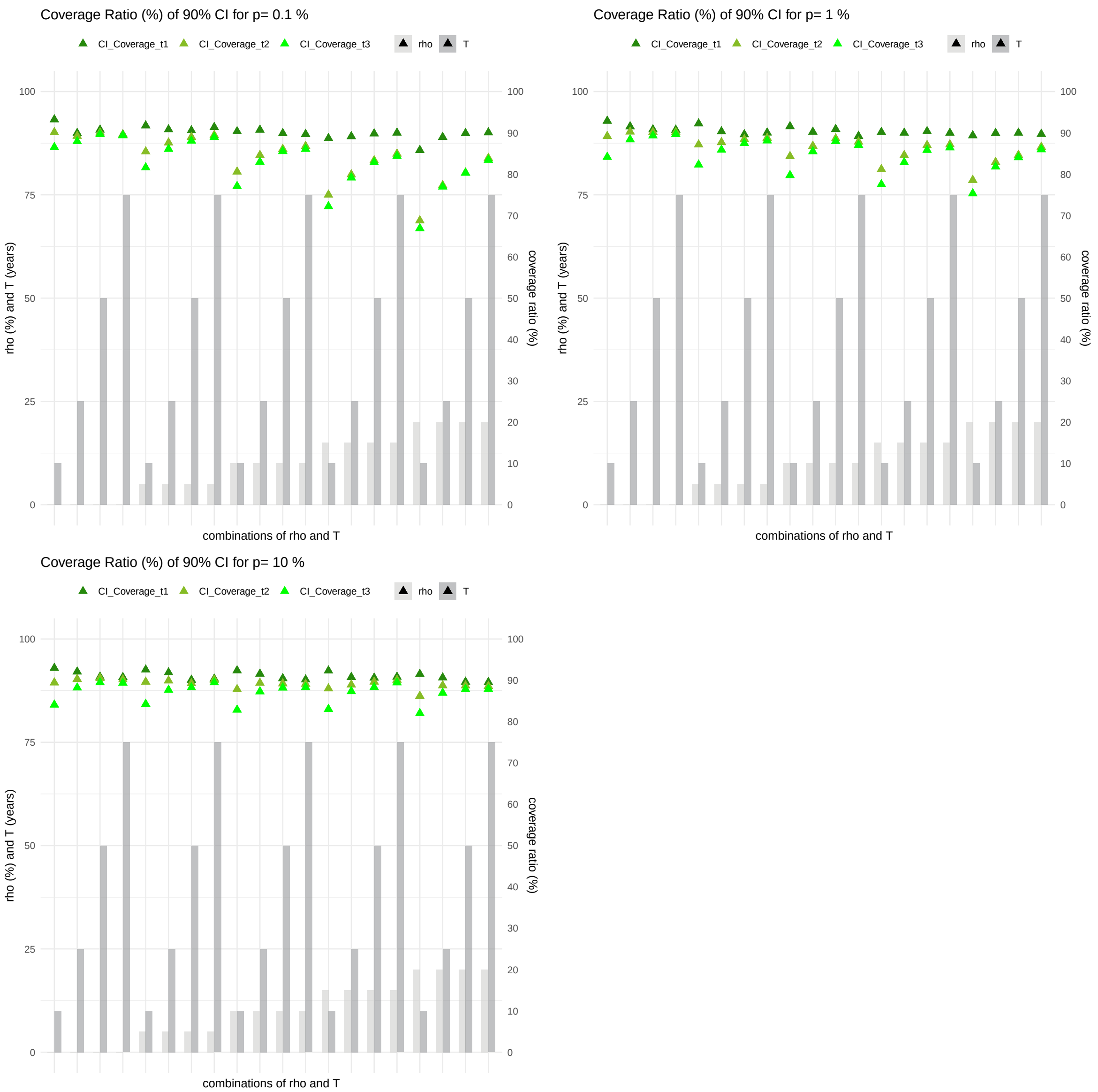

Figure 5: Coverage ratios (right y-axis) of 90% CI for Type I, II and III for different values of ρ and T (left y-axis). Top row: $p = 0.1\%$ (left), $p = 1\%$ (right). Bottom: $p = 10\%$.

Our key findings are:

- Type I generates good results for all model parameter values, however with the caveat (see above) that the asset correlation is treated as a known input.
- For Type II and III, the coverage ratios strongly decrease with increasing ρ and decreasing T. The higher the correlation among obligors, the longer the time series needs to be to achieve a coverage close to the nominal value. This interesting effect is more pronounced in low-default populations than in high-default populations.
- If T is interpreted as a number of non-overlapping 12-month periods, values around 10-15 periods typically observed in practice lead to actual coverage ratios well below 90% when defaults are correlated. Bootstrapping obviously does not improve the coverage ratios.
- Type II generally produces better results than Type III. However, as we will see below, this is due to our conservative use of the t distribution instead of the normal distribution in the CI derivation following eq. (14).

6.2 Detailed results for Type I

In the previous section, we observed that the Type I approach has the highest coverage ratio of CIs as well as the highest MoC C median levels, with the caveat that we needed to make assumptions about the correlation parameter. In this section and in Appendix A, we present our detailed results for the size and coverage ratio of Type I MoCs.

Figure 6 shows the 5th, 50th and 95th percentiles of the simulated MoC C values for our parameter set presented in section 6.1. The corresponding numbers can also be found in tables 2, 3 and 4 in Appendix A for $p = 0.1\%$, $p = 1\%$ and $p = 10\%$, respectively. In addition, these tables display:

- the coverage ratios of the two-sided CIs defined in eq. (7) and eq. (8) with $\alpha = 10\%$, i.e. 90% confidence level,
- the CI coverage ratio and MoC sizes in case ρ is to be estimated from the default rate time series by an MLE estimator, see at the end of this section.

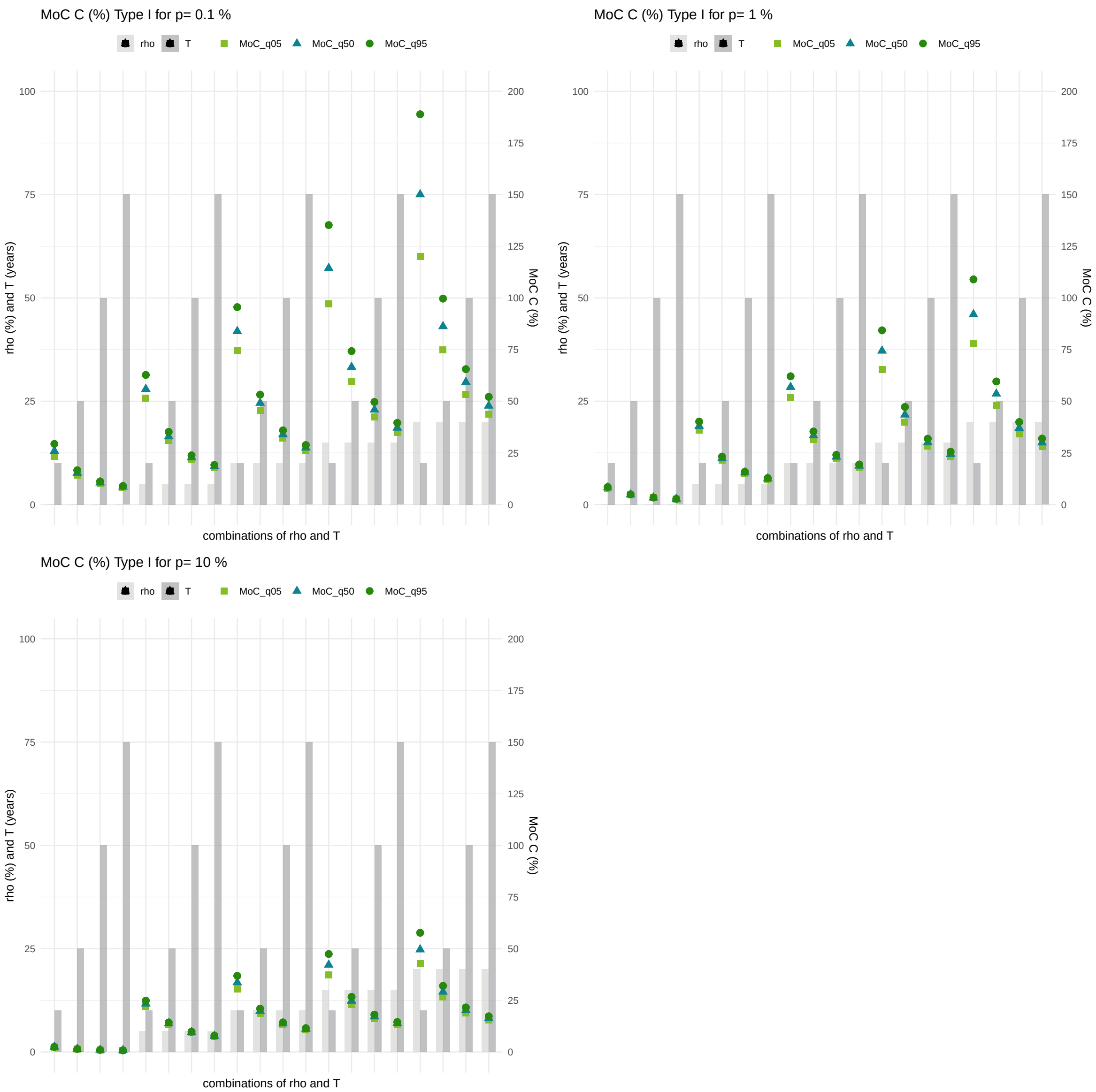

Figure 6: 5th, 50th and 95th percentiles of the simulated (relative) MoC C according to eq. (9) (right y-axis) for different values of ρ and T (left y-axis). Top row: $p = 0.1\%$ (left), $p = 1\%$ (right). Bottom: $p = 10\%$.

Comparing the results to fig. 4, we see that the dispersion of the MoC, e.g. the distance between the 5th and the 95th percentile for a certain parameter set, is small for $\rho = 0$ but grows significantly with increasing ρ and with decreasing T and p. In the extreme case of $p = 0.1\%$, $\rho = 0.2$ and $T = 10$, the result of a MoC calculation from one single observed default rate time series can vary randomly between 120.1% or 188.9%. This high dispersion explains the drop of the CI coverage ratio we observe in figure 5 and table 2: Less than 90% of the 5000 intervals we simulated contain the "true" PD value of 0.1% if the time series is too short. Under these conditions of low PD and high correlation, the time series should contain at least 25 observation periods if eq. (8) is used. Alternatively, the CI can be set to a higher value if a certain level of conservatism needs to be ensured.

We note that this randomness of the MoC result is even greater in practice since the input parameter ρ can only be estimated with a degree of uncertainty. As an example. in the last four columns of each of tables 2, 3 and 4 in Appendix A, we calculate how ρ would be estimated from our simulated time series using the Maximum Likelihood Estimator presented in [Düllmann and Gehde-Trapp, 2004,Tasche, 2008]. The coverage ratios resulting from this exercise are generally good for $p = 1\%$ and $p = 10\%$, but much lower than expected for the low-default case of $p = 0.1\%$.

6.3 Detailed results for Type II

In section 4, we discussed how the empirical variance of observed (e.g. annual) default rates can be used to quantify the LRADR uncertainty and pointed out that correlated default behaviour of obligors increases this empirical variance, which in itself is a random number, and unfortunately also makes it more volatile. We illustrated this effect in fig. 2. For our parameter set presented in section 6.1, we display the detailed numerical results regarding this behaviour and its consequences in terms of MoC and CI coverage ratio in tables 5, 6 and 7 in Appendix B.

In detail, these tables show

- the coverage ratio of the two-sided LRADR CIs defined in eq. (12) and eq. (13) with $\alpha = 10\%$, i.e. 90% confidence level,
- the 5th, 50th and 95th percentiles of the simulated relative Type II MoC C values from eq. (14),
- the theory value $\sigma_{R_L} = \sqrt{\mathrm{Var}(R_L)}$, i.e. the square root of eq. (5) and, for comparison, the square root of the mean of $\widehat{\mathrm{Var}(R_L)}$ from eq. (11) and
- the coverage ratio of the CI of the variance defined in eq. (16) with $\alpha = 10\%$, see footnote.

We find the Type II approach to be unbiased in the sense that the square root of the mean of $\widehat{\mathrm{Var}(R_L)}$ over our 5000 distributions is equal to the theoretical value $\sigma_{R_L} = \sqrt{\mathrm{Var}(R_L)}$ to a good approximation.[2]

Figure 7 shows the 5th, 50th and 95th percentiles of the simulated Type II MoC values for our parameter set presented in section 6.1.

[2] As a possible way to deal with the high volatility of $\widehat{\mathrm{Var}(R_L)}$ when the available time series is rather short, we test, in the last column of each table, a naive CI for $s^2_{r_D}$ inspired from the CI of the variance of a normally distributed variable, knowing that the distribution of $R_D(t)$ is in general not normal when defaults are correlated:

$$\widehat{\mathrm{Var}(R_L)}^{u,\alpha} = \frac{1}{T} \cdot \frac{s^2_{r_D} \cdot (T-1)}{\chi^2_{T-1,\alpha/2}}$$
$$\widehat{\mathrm{Var}(R_L)}^{l,\alpha} = \frac{1}{T} \cdot \frac{s^2_{r_D} \cdot (T-1)}{\chi^2_{T-1,1-\alpha/2}} \tag{16}$$

with the confidence level $\alpha = 10\%$. However, we find this approach to fail because it yields misleading CIs with a coverage ratio far below the nominal value of 90%.

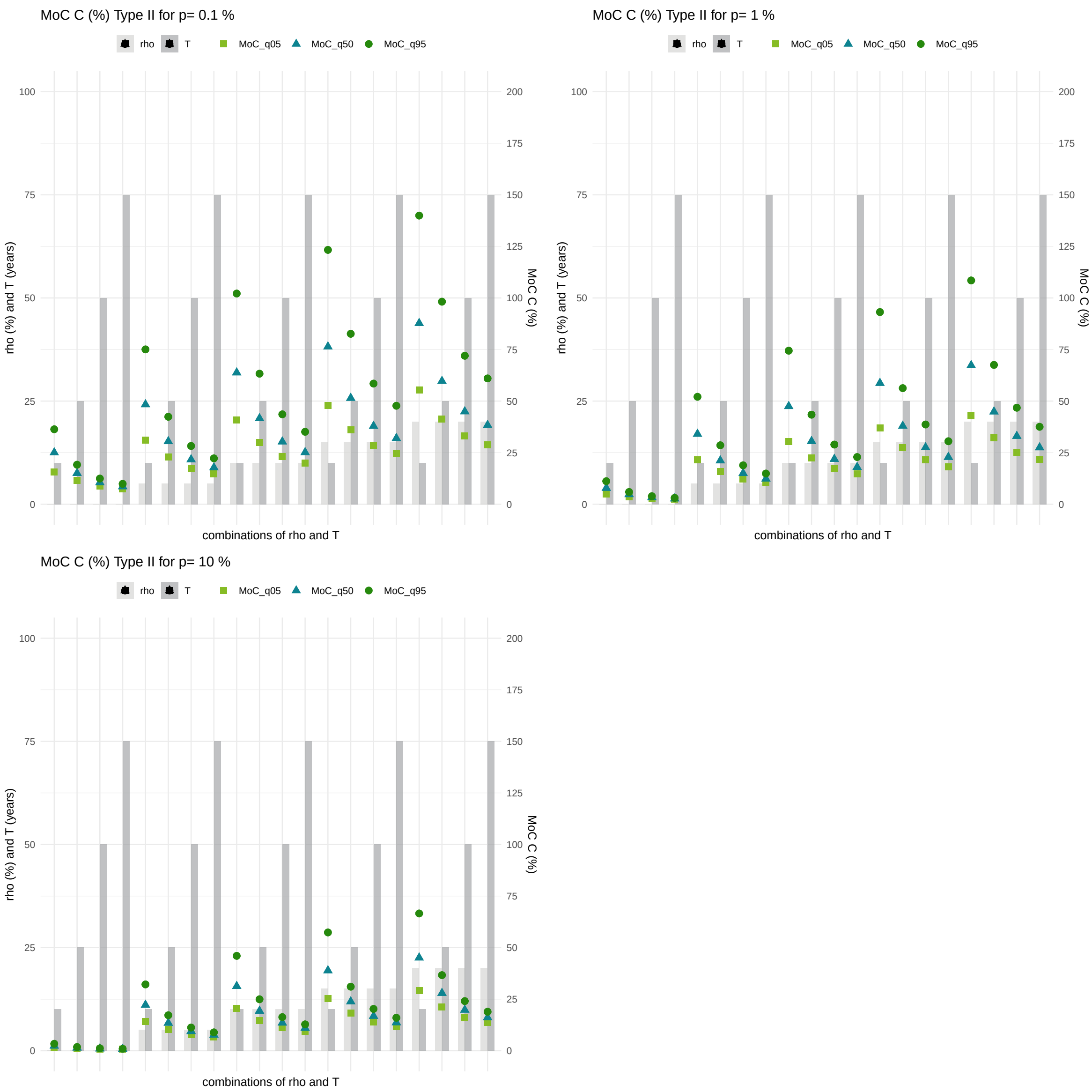

Figure 7: 5th, 50th and 95th percentiles of the simulated (relative) MoC C according to eq. (14) (right y-axis) for different values of ρ and T (left y-axis). Top row: $p = 0.1\%$ (left), $p = 1\%$ (right). Bottom: $p = 10\%$.

The patterns are very similar to the Type I case from figure 6, only with smaller values overall. We see that the dispersion of the MoC, e.g. the distance between the 5th and the 95th percentile for a certain parameter set, is small for $\rho = 0$, except for very small p, but grows significantly with increasing ρ and with decreasing T and p. In the extreme case of $p = 0.1\%$, $\rho = 0.2$ and $T = 10$, the result of a MoC calculation from one single observed default rate time series can vary randomly between 55.4% or 139.9%.[3] This high dispersion explains the drop of the CI coverage ratio we observe in figure 5 and table 5 if the time series is too short. Under these conditions of low PD and high correlation, the time series would need to contain more than 100 observation periods to allow for a coverage ratio close to the nominal one, which is unrealistic in practice for the case of annual default rates. Alternatively, the CI can be set to a higher value if a certain level of conservatism needs to be ensured.

6.4 Detailed results for Type III

For the Type III approach, we ran $n_{\text{boot}} = 1000$ bootstraps for each of the $n_{\text{sim}} = 5000$ simulations with our parameter set presented in section 6.1. We display the detailed numerical results in tables 8, 9 and

[3]Intuitively, we can explain this strong difference to Type I acknowledging that, with $N = 5000$ and $p = 0.1\%$, the expected number of defaults per observation period is only 5, hence at $\rho = 0.2$ the actual number is often zero, which lowers the empirical variance.

10 in Appendix C. In detail, these tables show

- the coverage ratio of the two-sided bootstrapped CI at 90% confidence level, i.e. the interval between the percentiles $P_5(r_L)$ and $P_{95}(r_L)$ and
- the 5th, 50th and 95th percentiles of the simulated relative Type III MoC values from eq. (15).

The latter are also represented graphically in figure 8.

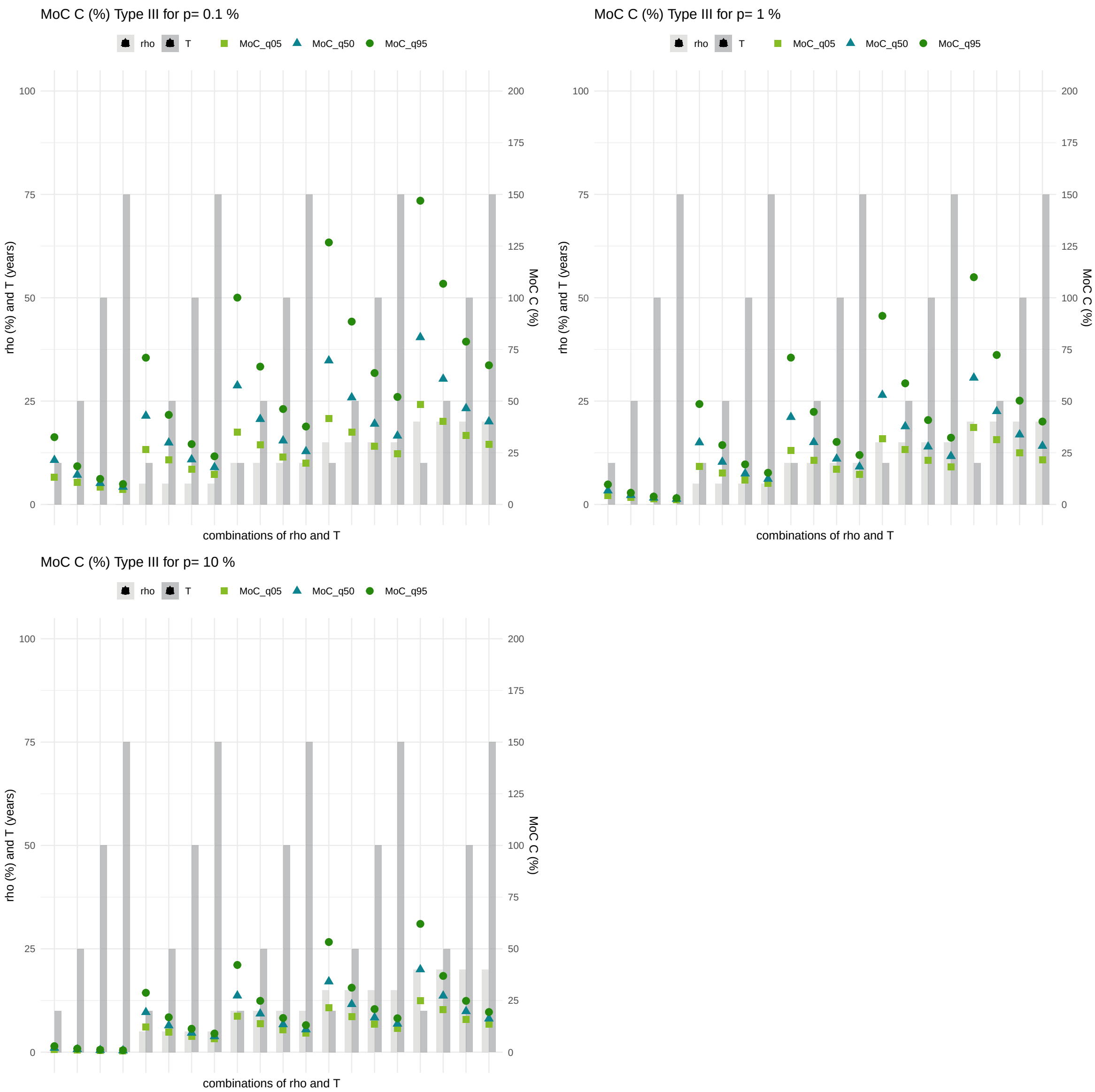

Figure 8: 5th, 50th and 95th percentiles of the simulated (relative) MoC C according to eq. (15)(right y-axis) for different values of ρ and T (left y-axis). Top row: $p = 0.1\%$ (left), $p = 1\%$ (right). Bottom: $p = 10\%$.

Comparing these tables and diagrams to the corresponding ones for the Type II approach, we see that the differences are small and mostly due to our conservative use of the t distribution in the Type II case instead of the normal distribution. Hence, the additional effort of bootstrapping from the observed default rate time series does not improve the accuracy of the MoCs for the problematic cases of low PD, high correlation and short time series. The only advantage of this approach is that bootstrapped confidence intervals for the LRADR are guaranteed to cover only positive values, independently of the input parameter set.

7 Conclusions

In this article we have analyzed, by means of an ASRF model simulation, three different approaches to quantify the statistical uncertainty of PD estimation:

- Type I: based on a distribution assumption for defaults,
- Type II: based on the empirical variance of default rates over time,
- Type III: based on bootstrapping from observed default rates.

We compared these approaches in terms of the resulting MoC levels and in terms of the coverage ratio of a 90% CI around the measured LRADR for different values of

- the length of the available time series,
- the PD level and
- the level of correlation between obligors.

We summarize the key results as follows: The dispersion of MoC results is small for uncorrelated defaults but grows significantly with increasing correlation and with decreasing PD level and time series length. As an extreme example, for the case of $p = 0.1\%$, $\rho = 0.2$ and $T = 10$ the 5th percentile of our Type I results is a 120.1% MoC relative to the LRADR and the 95th percentile is a 188.9% MoC. In these areas of the parameter space, we also observe that the actual confidence level of the MoC is lower than the expected (nominal) one. Types II and III on average lead to smaller MoCs[4] than Type I, but with even larger discrepancies between the actual and the nominal CL, entailing lower accuracy of the MoC. A caveat is that Type I relies on assumptions about the default distribution and correlation or on uncertain estimates of the correlation.

Table 1 summarizes the important advantages and disadvantages of the three types which should be considered from our point of view in the choice of a quantification approach for PD uncertainty.

	Advantages	Disadvantages
Type I	• good accuracy of CIs • low computational effort	• high median MoC results • model-dependent, correlation required as input parameter • CIs can reach into negative values
Type II	• model-independent, no correlation input required • low computational effort • low median MoC results	• volatile and possibly inaccurate CIs when data is scarce[5] and obligors are correlated • macroeconomic shocks can distort the MoC • CIs can reach into negative values
Type III	• model-independent, no correlation input required • low median MoC results • strictly positive CI boundaries	• volatile and possibly inaccurate CIs when data is scarce and obligors are correlated • macroeconomic shocks can distort the MoC • higher computational effort

Table 1: Summary of the advantages and disadvantages of the three types of approaches

[4]For Type II, the option exists to use eq. (13) for more conservatism instead of eq. (12), making the results larger than the Type III results.

Our results show that, in case of low correlation among obligors, the different approaches for the quantification of uncertainties yield rather similar results. In the case of correlated obligors, the uncertainty estimation itself is uncertain in the sense that either a model-dependent approach with a distribution assumption and an assumed (e.g. regulatory) or estimated strength of the correlation needs to be used or a very long time series must be available to produce reliable results based only on empirical data. In a low-default portfolio, or typical populations observed in practice with portfolio default rates of $\sim \mathcal{O}(1\%)$, the time series actually needs to be unrealistically long. Bootstrapping does not improve on this situation, therefore we are of the opinion that the additional computational effort of a Type III approach does not give it an advantage over a Type II approach.

An obvious solution to the issue of unreliability in situations with data scarcity and significant correlation is the use of high confidence levels to be on the safe side, which comes with the unwanted consequence of large conservative add-ons to the PD estimates. The use of a Type I approach with suitable - and sufficiently conservative - assumptions on the distribution of defaults is, from our point of view, an attractive alternative to Type II approaches in these situations. In a regulated environment, where the historical time window used for the estimation is not arbitrary but fixed by rules or an algorithm, a Type I approach with the use of a conditional variance as in eq. (4) constitutes an elegant solution.

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A Appendix: Simulation results for Type I approach (ASRF model)

ρ	T	Cov. ratio of 90% CI from eq. (7)	Cov. ratio of 90% CI from eq. (8)	q05 of Type I MoC C (rel.) from eq. (9)	q50 of Type I MoC C (rel.) from eq. (9)	q95 of Type I MoC C (rel.) from eq. (9)	Cov. ratio of 90% CI Type I with MLE estimates	q05 of Type I MoC C (rel.) with MLE estimates	q50 of Type I MoC C (rel.) with MLE estimates	q95 of Type I MoC C (rel.) with MLE estimates
0.00	10	90.8%	93.2%	23.3%	25.9%	29.3%	98.8%	29.7%	37.5%	50.0%
0.00	25	89.1%	89.9%	14.2%	15.3%	16.6%	98.6%	19.4%	23.1%	27.4%
0.00	50	89.8%	90.7%	10.1%	10.6%	11.2%	99.1%	14.3%	16.2%	18.3%
0.00	75	89.6%	89.6%	8.2%	8.6%	9.0%	98.7%	11.9%	13.2%	14.6%
0.00	100	90.5%	90.5%	7.2%	7.4%	7.7%	98.9%	10.4%	11.4%	12.4%
0.00	150	90.5%	90.5%	5.9%	6.0%	6.2%	98.7%	8.6%	9.3%	9.9%
0.00	200	89.6%	89.6%	5.1%	5.2%	5.4%	98.3%	7.5%	8.0%	8.5%
0.05	10	87.8%	91.8%	51.4%	55.9%	62.7%	90.8%	43.3%	57.4%	74.7%
0.05	25	89.5%	90.8%	31.1%	32.9%	35.2%	93.3%	29.6%	35.3%	41.7%
0.05	50	90.2%	90.6%	21.9%	22.8%	23.8%	93.9%	21.9%	24.7%	27.7%
0.05	75	90.9%	91.4%	17.9%	18.5%	19.2%	94.0%	18.3%	20.1%	22.1%
0.05	100	89.8%	90.2%	15.5%	16.0%	16.5%	93.6%	16.0%	17.4%	18.9%
0.05	150	89.9%	90.2%	12.7%	13.0%	13.3%	93.3%	13.3%	14.2%	15.2%
0.05	200	89.4%	89.5%	11.0%	11.2%	11.5%	93.3%	11.6%	12.3%	13.0%
0.10	10	87.1%	90.3%	74.7%	83.9%	95.5%	83.6%	50.9%	66.5%	90.3%
0.10	25	89.1%	90.7%	45.5%	49.1%	53.2%	86.1%	34.5%	41.3%	50.1%
0.10	50	89.2%	89.9%	32.1%	33.9%	35.9%	86.1%	25.5%	29.0%	33.1%
0.10	75	89.4%	89.7%	26.3%	27.5%	28.8%	85.7%	21.3%	23.7%	26.5%
0.10	100	89.1%	89.5%	22.8%	23.7%	24.7%	85.3%	18.7%	20.5%	22.5%
0.10	150	89.5%	89.8%	18.7%	19.3%	19.9%	84.7%	15.5%	16.7%	18.1%
0.10	200	89.9%	90.0%	16.2%	16.7%	17.2%	84.8%	13.6%	14.5%	15.5%
0.15	10	85.6%	88.7%	97.2%	114.4%	135.3%	74.9%	52.7%	71.7%	102.1%
0.15	25	88.5%	89.1%	59.7%	66.5%	74.3%	77.0%	36.3%	44.9%	56.0%
0.15	50	88.9%	89.8%	42.4%	45.9%	49.6%	75.5%	27.2%	31.7%	37.3%
0.15	75	89.6%	90.0%	34.8%	37.1%	39.5%	75.2%	22.7%	25.9%	29.5%
0.15	100	89.9%	90.1%	30.3%	32.0%	33.8%	73.8%	20.0%	22.4%	25.2%
0.15	150	90.1%	90.2%	24.9%	26.0%	27.2%	70.8%	16.7%	18.3%	20.1%
0.15	200	89.5%	89.7%	21.6%	22.5%	23.4%	68.8%	14.7%	15.9%	17.2%
0.20	10	83.9%	85.8%	120.1%	150.1%	188.9%	67.2%	51.1%	74.1%	111.7%
0.20	25	87.8%	89.0%	74.8%	86.2%	99.7%	69.3%	36.4%	47.2%	60.7%
0.20	50	89.4%	89.9%	53.3%	59.3%	65.5%	65.2%	27.8%	33.3%	40.2%
0.20	75	89.7%	90.1%	43.8%	47.8%	52.1%	64.4%	23.6%	27.4%	31.9%
0.20	100	89.6%	89.8%	38.1%	41.2%	44.4%	60.9%	20.7%	23.7%	27.0%
0.20	150	89.7%	89.8%	31.4%	33.5%	35.6%	56.0%	17.3%	19.4%	21.6%
0.20	200	90.1%	90.2%	27.4%	28.9%	30.5%	49.9%	15.3%	16.7%	18.4%

Table 2: Simulation of Type I MoC approach as described in section 3.2 for $p = 0.1\%$

ρ	T	Cov. ratio of 90% CI from eq. (7)	Cov. ratio of 90% CI from eq. (8)	q05 of Type I MoC (rel.) from eq. (9)	q50 of Type I MoC (rel.) from eq. (9)	q95 of Type I MoC (rel.) from eq. (9)	Cov. ratio of 90% CI Type I with MLE estimates	q05 of Type I MoC (rel.) with MLE estimates	q50 of Type I MoC (rel.) with MLE estimates	q95 of Type I MoC (rel.) with MLE estimates
0.00	10	89.5%	92.9%	7.9%	8.2%	8.5%	98.5%	9.5%	11.1%	13.6%
0.00	25	90.3%	91.5%	4.7%	4.8%	4.9%	98.3%	6.0%	6.7%	7.7%
0.00	50	90.3%	90.8%	3.3%	3.3%	3.4%	98.0%	4.3%	4.7%	5.2%
0.00	75	90.2%	90.7%	2.7%	2.7%	2.7%	98.2%	3.6%	3.8%	4.1%
0.00	100	90.4%	90.6%	2.3%	2.3%	2.4%	98.1%	3.1%	3.3%	3.5%
0.00	150	89.7%	89.8%	1.9%	1.9%	1.9%	98.3%	2.6%	2.7%	2.8%
0.00	200	90.1%	90.5%	1.6%	1.6%	1.7%	98.1%	2.2%	2.3%	2.4%
0.05	10	89.6%	92.2%	35.9%	38.0%	40.1%	89.4%	23.2%	36.0%	52.3%
0.05	25	89.1%	90.3%	21.6%	22.3%	23.2%	90.2%	17.4%	22.6%	28.8%
0.05	50	88.8%	89.6%	15.1%	15.5%	15.9%	91.2%	13.4%	15.9%	18.9%
0.05	75	89.6%	90.0%	12.3%	12.6%	12.8%	91.1%	11.2%	13.0%	15.0%
0.05	100	90.1%	90.3%	10.7%	10.8%	11.0%	91.5%	10.0%	11.3%	12.7%
0.05	150	90.6%	90.7%	8.7%	8.8%	9.0%	92.1%	8.3%	9.2%	10.2%
0.05	200	89.3%	89.5%	7.5%	7.6%	7.7%	91.1%	7.3%	8.0%	8.7%
0.10	10	88.7%	91.6%	51.9%	56.8%	62.1%	87.8%	33.1%	53.5%	78.8%
0.10	25	88.9%	90.2%	31.6%	33.4%	35.4%	90.3%	25.5%	33.6%	42.9%
0.10	50	90.3%	90.9%	22.2%	23.1%	24.0%	91.4%	19.6%	23.7%	28.0%
0.10	75	88.8%	89.2%	18.1%	18.7%	19.4%	91.1%	16.5%	19.3%	22.3%
0.10	100	90.4%	90.6%	15.7%	16.2%	16.6%	92.2%	14.6%	16.8%	18.8%
0.10	150	90.4%	90.6%	12.9%	13.2%	13.5%	92.2%	12.3%	13.7%	15.1%
0.10	200	89.5%	89.6%	11.1%	11.4%	11.6%	91.8%	10.8%	11.8%	12.9%
0.15	10	87.7%	90.2%	65.4%	74.4%	84.3%	86.2%	42.7%	69.2%	97.9%
0.15	25	88.8%	90.0%	40.0%	43.5%	47.2%	89.1%	32.9%	43.1%	53.6%
0.15	50	89.7%	90.4%	28.4%	30.1%	31.9%	91.0%	25.3%	30.4%	35.5%
0.15	75	89.6%	90.0%	23.2%	24.4%	25.6%	90.8%	21.3%	24.8%	28.1%
0.15	100	89.9%	90.3%	20.2%	21.0%	21.9%	91.8%	18.9%	21.5%	24.1%
0.15	150	90.0%	90.2%	16.6%	17.1%	17.7%	92.3%	15.8%	17.5%	19.2%
0.15	200	89.5%	89.7%	14.4%	14.8%	15.2%	92.0%	14.0%	15.2%	16.4%
0.20	10	86.8%	89.4%	77.8%	92.0%	109.0%	84.3%	52.8%	83.3%	111.6%
0.20	25	88.8%	89.9%	48.0%	53.6%	59.6%	88.1%	40.5%	51.6%	62.5%
0.20	50	89.3%	90.0%	34.2%	37.0%	39.9%	89.4%	30.8%	36.3%	41.5%
0.20	75	89.3%	89.7%	28.1%	30.0%	31.9%	89.5%	25.9%	29.5%	33.1%
0.20	100	90.2%	90.5%	24.4%	25.8%	27.2%	90.9%	22.9%	25.6%	28.3%
0.20	150	89.5%	89.7%	20.1%	21.0%	22.0%	89.6%	19.1%	20.9%	22.7%
0.20	200	89.4%	89.7%	17.4%	18.2%	18.9%	90.6%	16.7%	18.1%	19.4%

Table 3: Simulation of Type I MoC approach as described in section 3.2 for $p = 1\%$

ρ	T	Cov. ratio of 90% CI from eq. (7)	Cov. ratio of 90% CI from eq. (8)	q05 of Type I MoC C (rel.) from eq. (9)	q50 of Type I MoC C (rel.) from eq. (9)	q95 of Type I MoC C (rel.) from eq. (9)	Cov. ratio of 90% CI Type I with MLE estimates	q05 of Type I MoC C (rel.) with MLE estimates	q50 of Type I MoC C (rel.) with MLE estimates	q95 of Type I MoC C (rel.) with MLE estimates
0.00	10	89.6%	93.0%	2.4%	2.5%	2.5%	98.3%	2.8%	3.3%	4.0%
0.00	25	90.8%	92.1%	1.4%	1.5%	1.5%	98.3%	1.8%	2.0%	2.3%
0.00	50	90.2%	90.8%	1.0%	1.0%	1.0%	98.2%	1.3%	1.4%	1.5%
0.00	75	90.3%	90.7%	0.8%	0.8%	0.8%	98.0%	1.1%	1.2%	1.2%
0.00	100	90.2%	90.5%	0.7%	0.7%	0.7%	98.0%	0.9%	1.0%	1.1%
0.00	150	90.3%	90.3%	0.6%	0.6%	0.6%	98.2%	0.8%	0.8%	0.9%
0.00	200	90.0%	90.2%	0.5%	0.5%	0.5%	98.3%	0.7%	0.7%	0.7%
0.05	10	89.6%	92.6%	22.0%	23.4%	24.9%	88.6%	13.9%	21.7%	30.2%
0.05	25	90.5%	91.9%	13.3%	13.8%	14.3%	90.2%	10.4%	13.4%	16.6%
0.05	50	89.5%	90.1%	9.3%	9.6%	9.8%	89.5%	8.0%	9.4%	11.0%
0.05	75	90.1%	90.4%	7.6%	7.7%	7.9%	90.2%	6.7%	7.7%	8.7%
0.05	100	90.5%	90.9%	6.6%	6.7%	6.8%	90.6%	5.9%	6.7%	7.5%
0.05	150	89.7%	90.0%	5.4%	5.4%	5.5%	89.7%	5.0%	5.4%	6.0%
0.05	200	89.6%	89.6%	4.6%	4.7%	4.8%	89.8%	4.3%	4.7%	5.1%
0.10	10	88.8%	92.4%	30.6%	33.7%	36.9%	87.6%	20.3%	30.9%	42.7%
0.10	25	90.4%	91.6%	18.7%	19.8%	21.0%	89.5%	15.1%	19.3%	23.7%
0.10	50	89.6%	90.5%	13.2%	13.7%	14.3%	89.1%	11.5%	13.6%	15.8%
0.10	75	89.8%	90.2%	10.8%	11.1%	11.5%	89.5%	9.7%	11.0%	12.5%
0.10	100	90.4%	90.7%	9.3%	9.6%	9.9%	89.9%	8.6%	9.6%	10.6%
0.10	150	89.5%	89.7%	7.6%	7.8%	8.0%	89.5%	7.1%	7.8%	8.5%
0.10	200	90.3%	90.4%	6.6%	6.8%	6.9%	90.2%	6.3%	6.8%	7.3%
0.15	10	89.2%	92.3%	37.3%	42.1%	47.4%	87.8%	25.5%	38.9%	53.0%
0.15	25	89.5%	90.8%	22.9%	24.7%	26.7%	89.1%	19.0%	24.1%	29.3%
0.15	50	89.8%	90.6%	16.2%	17.1%	18.1%	89.9%	14.4%	16.9%	19.5%
0.15	75	90.6%	90.8%	13.3%	13.9%	14.5%	90.5%	12.1%	13.8%	15.5%
0.15	100	91.2%	91.5%	11.6%	12.0%	12.4%	91.1%	10.7%	11.9%	13.2%
0.15	150	90.5%	90.6%	9.5%	9.8%	10.1%	90.2%	8.9%	9.7%	10.6%
0.15	200	90.3%	90.5%	8.2%	8.4%	8.7%	90.3%	7.8%	8.4%	9.1%
0.20	10	88.5%	91.5%	42.8%	49.6%	57.7%	86.5%	30.0%	45.6%	61.8%
0.20	25	89.4%	90.6%	26.6%	29.1%	32.0%	89.2%	22.4%	28.3%	34.1%
0.20	50	89.1%	89.6%	18.9%	20.2%	21.6%	89.0%	17.1%	20.0%	22.8%
0.20	75	89.2%	89.5%	15.4%	16.4%	17.3%	89.4%	14.4%	16.3%	18.1%
0.20	100	90.3%	90.6%	13.5%	14.1%	14.8%	90.5%	12.7%	14.0%	15.5%
0.20	150	90.7%	90.9%	11.1%	11.5%	11.9%	90.3%	10.5%	11.5%	12.4%
0.20	200	89.7%	89.9%	9.6%	9.9%	10.3%	89.9%	9.2%	9.9%	10.6%

Table 4: Simulation of Type I MoC approach as described in section 3.2 for $p = 10\%$

B Appendix: Simulation results for Type II approach

ρ	T	Cov. ratio of 90% CI Type II from eq. (12)	Cov. ratio of 90% CI Type II from eq. (13)	q05 of Type II MoC C (rel.) from eq. (14)	q50 of Type II MoC C (rel.) from eq. (14)	q95 of Type II MoC C (rel.) from eq. (14)	σ_{R_L} (theory) from eq. (5)	sqrt of mean of $\widehat{\mathrm{Var}(R_L)}$ from eq. (11)	Cov. ratio of 90% CI for $\mathrm{Var}(R_L)$ from eq. (16)
0.00	10	86.6%	90.1%	15.6%	25.0%	36.3%	0.014%	0.014%	88.1%
0.00	25	88.0%	89.2%	11.5%	15.0%	19.1%	0.009%	0.009%	88.3%
0.00	50	89.5%	90.1%	8.7%	10.5%	12.4%	0.006%	0.006%	88.5%
0.00	75	89.2%	89.5%	7.4%	8.6%	9.9%	0.005%	0.005%	88.8%
0.00	100	90.3%	90.4%	6.5%	7.4%	8.3%	0.004%	0.004%	88.8%
0.00	150	90.2%	90.5%	5.4%	6.0%	6.6%	0.004%	0.004%	89.2%
0.00	200	89.5%	89.6%	4.8%	5.2%	5.7%	0.003%	0.003%	88.5%
0.05	10	82.3%	85.5%	31.0%	48.4%	75.1%	0.030%	0.030%	65.6%
0.05	25	86.4%	87.6%	22.8%	30.4%	42.4%	0.019%	0.019%	58.6%
0.05	50	88.3%	89.0%	17.4%	21.6%	28.2%	0.014%	0.014%	55.7%
0.05	75	88.9%	89.4%	14.7%	17.8%	22.2%	0.011%	0.011%	55.7%
0.05	100	88.5%	88.8%	13.2%	15.5%	19.1%	0.010%	0.010%	53.0%
0.05	150	88.8%	89.0%	11.0%	12.7%	15.1%	0.008%	0.008%	51.7%
0.05	200	88.8%	89.1%	9.7%	11.0%	12.9%	0.007%	0.007%	52.2%
0.10	10	77.5%	80.6%	40.8%	63.8%	102.1%	0.045%	0.044%	46.9%
0.10	25	83.1%	84.6%	29.9%	41.6%	63.3%	0.029%	0.029%	40.1%
0.10	50	85.6%	86.1%	23.2%	30.3%	43.6%	0.020%	0.020%	37.7%
0.10	75	86.1%	86.8%	20.0%	25.1%	35.1%	0.016%	0.017%	36.4%
0.10	100	86.8%	87.1%	18.0%	21.9%	29.9%	0.014%	0.014%	36.1%
0.10	150	87.8%	87.9%	15.2%	18.2%	23.8%	0.012%	0.012%	34.3%
0.10	200	88.2%	88.3%	13.6%	15.9%	20.1%	0.010%	0.010%	34.4%
0.15	10	72.3%	75.0%	47.8%	76.4%	123.3%	0.061%	0.062%	35.8%
0.15	25	78.9%	79.9%	36.1%	51.5%	82.6%	0.038%	0.039%	28.2%
0.15	50	82.8%	83.3%	28.2%	37.9%	58.5%	0.027%	0.027%	27.2%
0.15	75	84.7%	84.9%	24.5%	32.0%	47.7%	0.022%	0.022%	27.5%
0.15	100	85.3%	85.8%	22.1%	28.1%	40.9%	0.019%	0.019%	26.4%
0.15	150	86.9%	87.1%	19.0%	23.5%	32.9%	0.016%	0.016%	24.4%
0.15	200	87.0%	87.2%	16.9%	20.8%	28.7%	0.014%	0.014%	25.5%
0.20	10	66.4%	68.8%	55.4%	87.7%	139.9%	0.078%	0.076%	26.5%
0.20	25	76.3%	77.3%	41.2%	59.7%	98.2%	0.049%	0.049%	23.1%
0.20	50	79.7%	80.4%	33.1%	44.9%	72.0%	0.035%	0.034%	20.9%
0.20	75	83.5%	83.9%	28.8%	38.3%	61.0%	0.028%	0.029%	20.9%
0.20	100	83.6%	83.9%	25.8%	34.4%	53.7%	0.025%	0.025%	20.2%
0.20	150	85.3%	85.5%	22.3%	28.8%	43.6%	0.020%	0.020%	19.9%
0.20	200	85.7%	85.8%	20.2%	25.4%	37.2%	0.017%	0.017%	18.5%

Table 5: Simulation of Type II MoC approach as described in section 4 for $p = 0.1\%$

ρ	T	Cov. ratio of 90% CI Type II from eq. (12)	Cov. ratio of 90% CI Type II from eq. (13)	q05 of Type II MoC C (rel.) from eq. (14)	q50 of Type II MoC C (rel.) from eq. (14)	q95 of Type II MoC C (rel.) from eq. (14)	σ_{R_L} (theory) from eq. (5)	sqrt of mean of $\widehat{\mathrm{Var}(R_L)}$ from eq. (11)	Cov. ratio of 90% CI for $\mathrm{Var}(R_L)$ from eq. (16)
0.00	10	85.6%	89.2%	5.0%	7.8%	11.2%	0.044%	0.044%	90.3%
0.00	25	89.0%	90.3%	3.7%	4.8%	5.9%	0.028%	0.028%	90.5%
0.00	50	89.7%	90.2%	2.8%	3.3%	3.9%	0.020%	0.020%	89.4%
0.00	75	89.8%	90.2%	2.3%	2.7%	3.1%	0.016%	0.016%	89.9%
0.00	100	90.0%	90.4%	2.1%	2.3%	2.6%	0.014%	0.014%	89.9%
0.00	150	89.5%	89.7%	1.7%	1.9%	2.1%	0.011%	0.011%	89.8%
0.00	200	90.1%	90.2%	1.5%	1.6%	1.8%	0.010%	0.010%	90.1%
0.05	10	84.0%	87.1%	21.6%	34.0%	52.1%	0.206%	0.206%	74.5%
0.05	25	86.5%	87.7%	15.9%	21.1%	28.6%	0.131%	0.131%	68.3%
0.05	50	88.0%	88.5%	12.2%	15.0%	18.9%	0.092%	0.092%	66.7%
0.05	75	88.3%	88.8%	10.4%	12.3%	14.9%	0.075%	0.075%	65.9%
0.05	100	89.2%	89.5%	9.2%	10.6%	12.6%	0.065%	0.065%	65.6%
0.05	150	90.4%	90.6%	7.7%	8.7%	10.1%	0.053%	0.053%	64.2%
0.05	200	89.3%	89.3%	6.8%	7.6%	8.5%	0.046%	0.046%	64.0%
0.10	10	81.1%	84.3%	30.3%	47.5%	74.4%	0.308%	0.305%	61.7%
0.10	25	86.1%	86.8%	22.5%	30.5%	43.4%	0.195%	0.195%	54.9%
0.10	50	88.2%	88.6%	17.4%	21.9%	28.9%	0.138%	0.138%	52.8%
0.10	75	87.5%	87.8%	14.7%	18.0%	22.9%	0.112%	0.112%	49.5%
0.10	100	89.1%	89.3%	13.1%	15.7%	19.5%	0.097%	0.098%	49.9%
0.10	150	89.3%	89.5%	11.1%	12.8%	15.4%	0.079%	0.080%	48.4%
0.10	200	89.0%	89.1%	9.7%	11.2%	13.1%	0.069%	0.069%	48.4%
0.15	10	78.4%	81.2%	37.0%	58.7%	93.2%	0.400%	0.400%	50.2%
0.15	25	83.4%	84.6%	27.4%	37.9%	56.3%	0.253%	0.254%	45.2%
0.15	50	86.4%	87.0%	21.5%	27.5%	38.7%	0.179%	0.179%	42.5%
0.15	75	86.8%	87.2%	18.2%	22.8%	30.5%	0.146%	0.146%	40.5%
0.15	100	88.1%	88.3%	16.4%	19.9%	26.0%	0.126%	0.127%	41.6%
0.15	150	89.2%	89.5%	13.9%	16.4%	20.6%	0.103%	0.103%	40.5%
0.15	200	88.8%	88.9%	12.4%	14.4%	17.5%	0.089%	0.090%	39.6%
0.20	10	75.9%	78.6%	42.9%	67.3%	108.5%	0.491%	0.485%	43.0%
0.20	25	81.7%	82.9%	32.2%	44.9%	67.5%	0.310%	0.314%	37.8%
0.20	50	84.1%	84.6%	25.2%	33.0%	46.8%	0.219%	0.218%	35.4%
0.20	75	86.1%	86.6%	21.7%	27.5%	37.5%	0.179%	0.178%	36.5%
0.20	100	88.3%	88.5%	19.4%	24.1%	32.2%	0.155%	0.156%	34.3%
0.20	150	87.7%	87.8%	16.5%	19.9%	25.6%	0.127%	0.126%	33.8%
0.20	200	87.9%	88.1%	14.7%	17.4%	21.8%	0.110%	0.110%	33.0%

Table 6: Simulation of Type II MoC approach as described in section 4 for $p = 1\%$

ρ	T	Cov. ratio of 90% CI Type II from eq. (12)	Cov. ratio of 90% CI Type II from eq. (13)	q05 of Type II MoC C (rel.) from eq. (14)	q50 of Type II MoC C (rel.) from eq. (14)	q95 of Type II MoC C (rel.) from eq. (14)	σ_{R_L} (theory) from eq. (5)	sqrt of mean of $\widehat{\mathrm{Var}(R_L)}$ from eq. (11)	Cov. ratio of 90% CI for $\mathrm{Var}(R_L)$ from eq. (16)
0.00	10	86.0%	89.4%	1.5%	2.4%	3.4%	0.134%	0.134%	89.3%
0.00	25	89.0%	90.3%	1.1%	1.4%	1.8%	0.085%	0.085%	89.8%
0.00	50	89.8%	90.5%	0.8%	1.0%	1.2%	0.060%	0.060%	89.6%
0.00	75	89.9%	90.2%	0.7%	0.8%	0.9%	0.049%	0.049%	89.8%
0.00	100	89.5%	89.9%	0.6%	0.7%	0.8%	0.042%	0.042%	90.0%
0.00	150	89.9%	90.1%	0.5%	0.6%	0.6%	0.035%	0.035%	89.2%
0.00	200	89.9%	90.1%	0.5%	0.5%	0.5%	0.030%	0.030%	89.4%
0.05	10	86.2%	89.6%	14.3%	22.2%	32.2%	1.273%	1.276%	84.2%
0.05	25	88.4%	89.9%	10.3%	13.5%	17.2%	0.805%	0.806%	83.2%
0.05	50	88.8%	89.3%	7.9%	9.4%	11.2%	0.569%	0.569%	82.7%
0.05	75	89.8%	90.1%	6.6%	7.7%	8.9%	0.465%	0.466%	82.3%
0.05	100	90.1%	90.5%	5.9%	6.6%	7.5%	0.403%	0.403%	82.1%
0.05	150	89.5%	89.7%	4.9%	5.4%	6.0%	0.329%	0.329%	82.6%
0.05	200	89.3%	89.5%	4.3%	4.7%	5.1%	0.285%	0.285%	81.3%
0.10	10	84.5%	87.8%	20.5%	31.3%	46.0%	1.831%	1.836%	80.2%
0.10	25	88.3%	89.4%	14.7%	19.3%	24.9%	1.158%	1.163%	76.9%
0.10	50	88.5%	89.2%	11.3%	13.5%	16.3%	0.819%	0.819%	76.8%
0.10	75	88.8%	89.1%	9.4%	11.0%	12.8%	0.669%	0.669%	75.9%
0.10	100	89.6%	89.9%	8.4%	9.5%	10.9%	0.579%	0.578%	76.2%
0.10	150	89.2%	89.5%	7.0%	7.8%	8.7%	0.473%	0.473%	75.4%
0.10	200	89.9%	90.0%	6.1%	6.7%	7.4%	0.409%	0.409%	74.9%
0.15	10	85.0%	88.0%	25.2%	38.9%	57.3%	2.284%	2.294%	77.0%
0.15	25	87.7%	88.9%	18.2%	23.8%	31.0%	1.444%	1.447%	73.2%
0.15	50	89.1%	89.6%	13.9%	16.8%	20.3%	1.021%	1.021%	71.9%
0.15	75	89.7%	90.1%	11.7%	13.7%	16.0%	0.834%	0.833%	73.1%
0.15	100	90.6%	90.9%	10.3%	11.8%	13.6%	0.722%	0.721%	71.6%
0.15	150	89.8%	90.0%	8.7%	9.7%	10.9%	0.590%	0.590%	70.4%
0.15	200	89.9%	90.1%	7.6%	8.4%	9.3%	0.511%	0.511%	70.2%
0.20	10	83.3%	86.2%	29.3%	45.1%	66.5%	2.686%	2.685%	72.9%
0.20	25	87.6%	88.7%	21.3%	27.9%	36.7%	1.699%	1.705%	68.3%
0.20	50	88.2%	88.7%	16.3%	19.8%	24.1%	1.201%	1.202%	67.5%
0.20	75	88.3%	88.6%	13.8%	16.1%	18.9%	0.981%	0.984%	68.0%
0.20	100	89.6%	90.0%	12.2%	14.0%	16.0%	0.849%	0.851%	68.4%
0.20	150	90.0%	90.2%	10.2%	11.4%	12.8%	0.693%	0.692%	67.6%
0.20	200	89.3%	89.5%	8.9%	9.8%	10.9%	0.601%	0.599%	65.7%

Table 7: Simulation of Type II MoC approach as described in section 4 for $p = 10\%$

C Appendix: Simulation results for Type III approach

ρ	T	Cov. ratio of 90% CI Type III	q05 of Type III MoC C (rel.)	q50 of Type III MoC C (rel.)	q95 of Type III MoC C (rel.)
0.00	10	86.5%	13.2%	21.4%	32.6%
0.00	25	87.9%	10.8%	14.3%	18.5%
0.00	50	89.7%	8.4%	10.3%	12.4%
0.00	75	89.4%	7.2%	8.4%	9.9%
0.00	100	90.3%	6.3%	7.3%	8.4%
0.00	150	90.1%	5.3%	6.0%	6.7%
0.00	200	89.7%	4.6%	5.2%	5.8%
0.05	10	81.6%	26.5%	42.9%	71.0%
0.05	25	86.1%	21.5%	29.9%	43.3%
0.05	50	88.1%	17.0%	21.7%	29.2%
0.05	75	89.0%	14.5%	17.9%	23.3%
0.05	100	88.6%	13.0%	15.7%	19.8%
0.05	150	88.8%	10.9%	12.9%	15.8%
0.05	200	88.8%	9.6%	11.2%	13.3%
0.10	10	77.1%	35.1%	57.5%	100.1%
0.10	25	83.0%	28.7%	41.3%	66.7%
0.10	50	85.5%	22.8%	30.9%	46.2%
0.10	75	86.1%	20.0%	25.7%	37.7%
0.10	100	86.6%	17.9%	22.6%	31.8%
0.10	150	87.5%	15.1%	18.6%	25.3%
0.10	200	88.2%	13.6%	16.3%	21.2%
0.15	10	72.2%	41.5%	69.6%	126.7%
0.15	25	79.2%	34.9%	51.7%	88.5%
0.15	50	82.9%	28.1%	39.0%	63.6%
0.15	75	84.4%	24.5%	33.2%	52.0%
0.15	100	85.8%	22.3%	29.2%	44.6%
0.15	150	87.0%	19.0%	24.4%	35.7%
0.15	200	86.9%	17.0%	21.5%	31.1%
0.20	10	66.9%	48.3%	80.8%	146.9%
0.20	25	77.0%	40.2%	60.7%	106.8%
0.20	50	80.3%	33.3%	46.5%	78.8%
0.20	75	83.4%	29.1%	40.1%	67.3%
0.20	100	83.9%	26.1%	35.9%	59.3%
0.20	150	85.2%	22.6%	30.2%	47.3%
0.20	200	85.8%	20.6%	26.5%	40.4%

Table 8: Simulation of Type III MoC approach as described in section 5 for $p = 0.1\%$

ρ	T	Cov. ratio of 90% CI Type III	q05 of Type III MoC C (rel.)	q50 of Type III MoC C (rel.)	q95 of Type III MoC C (rel.)
0.00	10	84.1%	4.2%	6.7%	9.7%
0.00	25	88.4%	3.4%	4.5%	5.7%
0.00	50	89.3%	2.6%	3.2%	3.8%
0.00	75	89.7%	2.3%	2.6%	3.1%
0.00	100	89.7%	2.0%	2.3%	2.6%
0.00	150	89.3%	1.7%	1.9%	2.1%
0.00	200	89.9%	1.5%	1.6%	1.8%
0.05	10	82.3%	18.3%	30.0%	48.6%
0.05	25	85.9%	15.1%	20.6%	28.7%
0.05	50	87.5%	11.9%	14.9%	19.4%
0.05	75	88.1%	10.2%	12.3%	15.3%
0.05	100	89.0%	9.0%	10.7%	13.0%
0.05	150	90.2%	7.6%	8.8%	10.4%
0.05	200	89.1%	6.7%	7.6%	8.8%
0.10	10	79.7%	26.1%	42.2%	71.0%
0.10	25	85.5%	21.4%	30.1%	44.8%
0.10	50	88.0%	17.1%	22.1%	30.3%
0.10	75	87.1%	14.5%	18.2%	23.9%
0.10	100	88.8%	13.0%	15.9%	20.3%
0.10	150	89.0%	11.0%	13.0%	16.2%
0.10	200	88.6%	9.6%	11.3%	13.7%
0.15	10	77.5%	31.8%	52.9%	91.2%
0.15	25	82.9%	26.5%	37.7%	58.6%
0.15	50	85.8%	21.2%	27.9%	40.8%
0.15	75	86.4%	18.1%	23.4%	32.3%
0.15	100	87.6%	16.4%	20.5%	27.4%
0.15	150	88.9%	13.9%	16.8%	21.8%
0.15	200	88.5%	12.4%	14.7%	18.4%
0.20	10	75.3%	37.3%	61.3%	109.9%
0.20	25	81.8%	31.2%	45.0%	72.3%
0.20	50	84.1%	25.1%	33.8%	50.2%
0.20	75	86.0%	21.6%	28.3%	40.1%
0.20	100	87.5%	19.3%	24.9%	34.6%
0.20	150	87.2%	16.6%	20.5%	27.3%
0.20	200	87.7%	14.8%	17.9%	23.1%

Table 9: Simulation of Type III MoC approach as described in section 5 for $p = 1\%$

ρ	T	Cov. ratio of 90% CI Type III	q05 of Type III MoC C (rel.)	q50 of Type III MoC C (rel.)	q95 of Type III MoC C (rel.)
0.00	10	84.1%	1.2%	2.0%	2.9%
0.00	25	88.2%	1.0%	1.3%	1.7%
0.00	50	89.5%	0.8%	1.0%	1.1%
0.00	75	89.4%	0.7%	0.8%	0.9%
0.00	100	89.4%	0.6%	0.7%	0.8%
0.00	150	89.8%	0.5%	0.6%	0.6%
0.00	200	89.7%	0.4%	0.5%	0.5%
0.05	10	84.3%	12.1%	19.3%	28.7%
0.05	25	87.7%	9.7%	12.9%	16.9%
0.05	50	88.2%	7.6%	9.3%	11.3%
0.05	75	89.5%	6.5%	7.6%	9.0%
0.05	100	89.8%	5.7%	6.6%	7.6%
0.05	150	89.1%	4.8%	5.4%	6.1%
0.05	200	89.3%	4.2%	4.7%	5.3%
0.10	10	82.9%	17.4%	27.3%	42.2%
0.10	25	87.3%	13.8%	18.6%	24.8%
0.10	50	88.2%	10.9%	13.4%	16.5%
0.10	75	88.3%	9.2%	11.0%	13.1%
0.10	100	89.2%	8.2%	9.5%	11.1%
0.10	150	89.1%	6.8%	7.8%	8.9%
0.10	200	89.7%	6.0%	6.7%	7.6%
0.15	10	83.0%	21.5%	34.2%	53.3%
0.15	25	87.3%	17.3%	23.1%	31.2%
0.15	50	88.3%	13.5%	16.7%	20.8%
0.15	75	89.5%	11.5%	13.7%	16.4%
0.15	100	90.3%	10.2%	11.9%	14.0%
0.15	150	89.7%	8.5%	9.8%	11.2%
0.15	200	89.8%	7.5%	8.4%	9.5%
0.20	10	82.0%	24.9%	40.0%	62.0%
0.20	25	86.9%	20.5%	27.3%	36.9%
0.20	50	87.8%	15.9%	19.7%	24.8%
0.20	75	87.9%	13.6%	16.2%	19.4%
0.20	100	89.3%	12.0%	14.1%	16.5%
0.20	150	89.9%	10.0%	11.5%	13.2%
0.20	200	89.1%	8.8%	9.9%	11.3%

Table 10: Simulation of Type III MoC approach as described in section 5 for $p = 10\%$

D Appendix: Model-independent estimator for the unconditional variance of the LRADR

In section 3, we have described a model-dependent approach to obtain the variance of the LRADR taking correlations among obligors into account. In eq. (7) of [Wosnitza, 2023], a model-independent estimator for this variance,

$$\widehat{\mathrm{Var}(R_L)} = \frac{1}{T^2} \cdot \frac{\tilde{\rho}^D}{1-\tilde{\rho}^D} \cdot \sum_{t=1}^{T} r_D(t)(1-r_D(t)), \tag{17}$$

with $\tilde{\rho}^D = \frac{1+(N-1)\rho^D}{N}$ and ρ^D being a default correlation parameter, has been derived under the following assumptions:

- finite and constant number of obligors $N(t) = N$ at each point in time t
- uniform PD for all obligors at a given time t
- no autocorrelation of the (conditional) PDs at different points in time
- uniform and constant pairwise default correlation ρ^D between all obligors.

The second assumption could be interpreted in terms of a PD conditional on the state of the economy at t, however if the assumption holds at all times t it automatically holds for the LRADR as well. We note that the ASRF model is a realization of these assumptions, with the uniform unconditional PD p, uniform conditional PDs $R_D(t)$ and uniform default correlation, see e.g. page 241 of [Baesens et al., 2016],

$$\rho^D = \frac{\Phi_2\left(\Phi^{-1}(p), \Phi^{-1}(p); \rho\right) - p^2}{p(1-p)}. \tag{18}$$

With this expression, one can easily confirm the equivalence of the pre-final result eq. (8) on page 40 of [Wosnitza, 2023] to our formulas for the ASRF model. We obtain the unconditional variance expression in our eq. (5) if, in eq. (8) on page 40 of [Wosnitza, 2023], PD_t is interpreted as an unconditional expectation value of the default rate at t, hence $PD_t = p$. Alternatively, we obtain the conditional variance in eq. (4) if PD_t is interpreted as the conditional expectation value $\mathrm{E}_c(R_D(t))$ and ρ_D is set to zero since, conditional on the systematic factor, defaults are uncorrelated in the ASRF model. The author then transforms this result to the expression (17) above in order to estimate $\mathrm{Var}(R_L)$ from the observed $r_D(t)$.[6]

Comparing eq. (17) to eq. (5), we find that eq. (17) is model-independent in the sense that it takes the observed $r_D(t)$ as inputs directly instead of the ASRF parameter p which is estimated by r_L in practice. Moreover, the default correlation ρ^D is required instead of the ASRF model's asset correlation ρ. While ρ_D needs to be estimated from data, ρ can either be estimated from data or one can rely on the formulas for ρ prescribed in the Basel standards.

The approach of [Wosnitza, 2023] is neither a pure Type I nor a pure Type II approach to MoC C in our typology. Like Type II approaches, it is model-independent and based on the observed default rates, however not on their empirical variance over time. Like Type I approaches, it requires some basic assumptions about the obligor PDs and uses a correlation parameter as an input.

In the tables 11, 12 and 13 of this Appendix, using the 5000 simulation runs presented in section 6, we display the 5th, 50th and 95th percentiles of the square root of $\widehat{\mathrm{Var}(R_L)}$ obtained from eq. (17) using the ASRF default correlation, i.e. inserting the simulated r_L into eq. (18) as estimator for p. Comparing these values to the same percentiles of the square root of $\widehat{\mathrm{Var}(R_L)}$ obtained inserting the simulated r_L into eq. (5) as estimator for p, we find that they are equal to a good approximation, which confirms the equivalence of the two approaches in case the default correlation ρ^D is related to the ASRF model's asset correlation via eq. (18). For comparison, we also display the theoretical value of σ_{R_L} from eq. (5) using p as an input parameter and the sample standard deviation s_{r_L} of the 5000 simulated LRADR values. We find that both values are close to the median values of $\widehat{\mathrm{Var}(R_L)}$.

[6] In the last step of this transformation, $1/T \sum_t \mathrm{E}(DR_t(1-DR_t))$ is replaced by the observed $1/T \sum_t (DR_t(1-DR_t))$, which is only possible if $1/T \sum_t \mathrm{E}(DR_t(1-DR_t)) = \mathrm{E}(DR(1-DR))$, i.e. it exists a constant uniform expectation value for $DR(1-DR)$, independent of the state of the economy, which can be estimated by an average of the measurements at different points in time.

ρ	T	σ_{R_L} (theory) from eq. (5)	s_{r_L} in 5000 sim.	q05 of σ_{R_L} from eq. (5) with $p = r_L$	q50 of σ_{R_L} from eq. (5) with $p = r_L$	q95 of σ_{R_L} from eq. (5) with $p = r_L$	q05 of σ_{R_L} from eq. (17) with $p = r_L$	q50 of σ_{R_L} from eq. (17) with $p = r_L$	q95 of σ_{R_L} from eq. (17) with $p = r_L$
0.00	10	0.014%	0.014%	0.012%	0.014%	0.016%	0.012%	0.014%	0.016%
0.00	25	0.009%	0.009%	0.008%	0.009%	0.010%	0.008%	0.009%	0.010%
0.00	50	0.006%	0.006%	0.006%	0.006%	0.007%	0.006%	0.006%	0.007%
0.00	75	0.005%	0.005%	0.005%	0.005%	0.005%	0.005%	0.005%	0.005%
0.00	100	0.004%	0.004%	0.004%	0.004%	0.005%	0.004%	0.004%	0.005%
0.00	150	0.004%	0.004%	0.004%	0.004%	0.004%	0.004%	0.004%	0.004%
0.00	200	0.003%	0.003%	0.003%	0.003%	0.003%	0.003%	0.003%	0.003%
0.05	10	0.030%	0.031%	0.019%	0.030%	0.043%	0.019%	0.030%	0.043%
0.05	25	0.019%	0.019%	0.015%	0.019%	0.025%	0.015%	0.019%	0.025%
0.05	50	0.014%	0.013%	0.011%	0.013%	0.016%	0.011%	0.013%	0.016%
0.05	75	0.011%	0.011%	0.010%	0.011%	0.013%	0.010%	0.011%	0.013%
0.05	100	0.010%	0.010%	0.008%	0.010%	0.011%	0.008%	0.010%	0.011%
0.05	150	0.008%	0.008%	0.007%	0.008%	0.009%	0.007%	0.008%	0.009%
0.05	200	0.007%	0.007%	0.006%	0.007%	0.007%	0.006%	0.007%	0.007%
0.10	10	0.045%	0.045%	0.023%	0.042%	0.076%	0.023%	0.042%	0.076%
0.10	25	0.029%	0.029%	0.019%	0.028%	0.041%	0.019%	0.028%	0.041%
0.10	50	0.020%	0.020%	0.015%	0.020%	0.026%	0.015%	0.020%	0.026%
0.10	75	0.016%	0.017%	0.013%	0.016%	0.020%	0.013%	0.016%	0.020%
0.10	100	0.014%	0.014%	0.012%	0.014%	0.017%	0.012%	0.014%	0.017%
0.10	150	0.012%	0.012%	0.010%	0.012%	0.013%	0.010%	0.012%	0.013%
0.10	200	0.010%	0.010%	0.009%	0.010%	0.012%	0.009%	0.010%	0.012%
0.15	10	0.061%	0.063%	0.025%	0.054%	0.116%	0.025%	0.054%	0.116%
0.15	25	0.038%	0.039%	0.022%	0.036%	0.061%	0.022%	0.036%	0.060%
0.15	50	0.027%	0.027%	0.018%	0.026%	0.038%	0.018%	0.026%	0.038%
0.15	75	0.022%	0.022%	0.016%	0.022%	0.029%	0.016%	0.022%	0.029%
0.15	100	0.019%	0.019%	0.015%	0.019%	0.025%	0.015%	0.019%	0.025%
0.15	150	0.016%	0.016%	0.013%	0.015%	0.019%	0.013%	0.015%	0.019%
0.15	200	0.014%	0.014%	0.011%	0.013%	0.016%	0.011%	0.013%	0.016%
0.20	10	0.078%	0.077%	0.025%	0.064%	0.161%	0.025%	0.064%	0.161%
0.20	25	0.049%	0.049%	0.025%	0.045%	0.082%	0.025%	0.045%	0.081%
0.20	50	0.035%	0.034%	0.022%	0.033%	0.052%	0.022%	0.033%	0.052%
0.20	75	0.028%	0.028%	0.019%	0.028%	0.040%	0.019%	0.028%	0.040%
0.20	100	0.025%	0.024%	0.018%	0.024%	0.033%	0.018%	0.024%	0.033%
0.20	150	0.020%	0.020%	0.015%	0.020%	0.026%	0.015%	0.020%	0.026%
0.20	200	0.017%	0.017%	0.014%	0.017%	0.022%	0.014%	0.017%	0.022%

Table 11: Simulation of LRADR standard deviation and different calculation approaches for $p = 0.1\%$

ρ	T	σ_{R_L} (theory) from eq. (5)	s_{r_L} in 5000 sim.	q05 of σ_{R_L} from eq. (5) with $p = r_L$	q50 of σ_{R_L} from eq. (5) with $p = r_L$	q95 of σ_{R_L} from eq. (5) with $p = r_L$	q05 of σ_{R_L} from eq. (17) with $p = r_L$	q50 of σ_{R_L} from eq. (17) with $p = r_L$	q95 of σ_{R_L} from eq. (17) with $p = r_L$
0.00	10	0.044%	0.045%	0.043%	0.044%	0.046%	0.043%	0.044%	0.046%
0.00	25	0.028%	0.028%	0.027%	0.028%	0.029%	0.028%	0.028%	0.029%
0.00	50	0.020%	0.020%	0.020%	0.020%	0.020%	0.020%	0.020%	0.020%
0.00	75	0.016%	0.016%	0.016%	0.016%	0.016%	0.016%	0.016%	0.016%
0.00	100	0.014%	0.014%	0.014%	0.014%	0.014%	0.014%	0.014%	0.014%
0.00	150	0.011%	0.012%	0.011%	0.011%	0.012%	0.011%	0.011%	0.012%
0.00	200	0.010%	0.010%	0.010%	0.010%	0.010%	0.010%	0.010%	0.010%
0.05	10	0.206%	0.203%	0.152%	0.203%	0.266%	0.152%	0.203%	0.266%
0.05	25	0.131%	0.133%	0.108%	0.130%	0.155%	0.108%	0.130%	0.155%
0.05	50	0.092%	0.093%	0.081%	0.092%	0.104%	0.081%	0.092%	0.104%
0.05	75	0.075%	0.075%	0.068%	0.075%	0.083%	0.068%	0.075%	0.083%
0.05	100	0.065%	0.065%	0.059%	0.065%	0.071%	0.060%	0.065%	0.071%
0.05	150	0.053%	0.052%	0.049%	0.053%	0.057%	0.050%	0.053%	0.057%
0.05	200	0.046%	0.047%	0.043%	0.046%	0.049%	0.043%	0.046%	0.049%
0.10	10	0.308%	0.305%	0.194%	0.296%	0.445%	0.194%	0.297%	0.445%
0.10	25	0.195%	0.194%	0.146%	0.192%	0.246%	0.146%	0.192%	0.246%
0.10	50	0.138%	0.136%	0.113%	0.137%	0.164%	0.113%	0.137%	0.164%
0.10	75	0.112%	0.114%	0.096%	0.112%	0.130%	0.096%	0.112%	0.130%
0.10	100	0.097%	0.097%	0.085%	0.097%	0.110%	0.085%	0.097%	0.110%
0.10	150	0.079%	0.079%	0.071%	0.079%	0.088%	0.071%	0.079%	0.088%
0.10	200	0.069%	0.069%	0.062%	0.069%	0.075%	0.063%	0.069%	0.075%
0.15	10	0.400%	0.402%	0.221%	0.377%	0.626%	0.221%	0.378%	0.625%
0.15	25	0.253%	0.258%	0.176%	0.248%	0.346%	0.176%	0.249%	0.345%
0.15	50	0.179%	0.180%	0.139%	0.177%	0.222%	0.140%	0.177%	0.222%
0.15	75	0.146%	0.147%	0.119%	0.145%	0.176%	0.119%	0.145%	0.175%
0.15	100	0.126%	0.127%	0.107%	0.126%	0.149%	0.107%	0.126%	0.148%
0.15	150	0.103%	0.103%	0.090%	0.103%	0.118%	0.090%	0.103%	0.118%
0.15	200	0.089%	0.090%	0.079%	0.089%	0.100%	0.079%	0.089%	0.100%
0.20	10	0.491%	0.480%	0.238%	0.452%	0.807%	0.239%	0.455%	0.805%
0.20	25	0.310%	0.315%	0.203%	0.300%	0.442%	0.203%	0.301%	0.441%
0.20	50	0.219%	0.221%	0.163%	0.215%	0.284%	0.163%	0.215%	0.283%
0.20	75	0.179%	0.179%	0.140%	0.177%	0.221%	0.140%	0.177%	0.220%
0.20	100	0.155%	0.154%	0.127%	0.154%	0.188%	0.127%	0.154%	0.187%
0.20	150	0.127%	0.127%	0.106%	0.126%	0.148%	0.107%	0.126%	0.148%
0.20	200	0.110%	0.111%	0.095%	0.109%	0.126%	0.095%	0.109%	0.126%

Table 12: Simulation of LRADR standard deviation and different calculation approaches for $p = 1\%$

ρ	T	σ_{R_L} (theory) from eq. (5)	s_{r_L} in 5000 sim.	q05 of σ_{R_L} from eq. (5) with $p = r_L$	q50 of σ_{R_L} from eq. (5) with $p = r_L$	q95 of σ_{R_L} from eq. (5) with $p = r_L$	q05 of σ_{R_L} from eq. (17) with $p = r_L$	q50 of σ_{R_L} from eq. (17) with $p = r_L$	q95 of σ_{R_L} from eq. (17) with $p = r_L$
0.00	10	0.134%	0.136%	0.133%	0.134%	0.136%	0.133%	0.134%	0.136%
0.00	25	0.085%	0.084%	0.084%	0.085%	0.085%	0.084%	0.085%	0.085%
0.00	50	0.060%	0.059%	0.060%	0.060%	0.060%	0.060%	0.060%	0.060%
0.00	75	0.049%	0.049%	0.049%	0.049%	0.049%	0.049%	0.049%	0.049%
0.00	100	0.042%	0.042%	0.042%	0.042%	0.043%	0.042%	0.042%	0.043%
0.00	150	0.035%	0.034%	0.035%	0.035%	0.035%	0.035%	0.035%	0.035%
0.00	200	0.030%	0.030%	0.030%	0.030%	0.030%	0.030%	0.030%	0.030%
0.05	10	1.273%	1.265%	1.081%	1.267%	1.458%	1.083%	1.267%	1.457%
0.05	25	0.805%	0.789%	0.732%	0.805%	0.880%	0.733%	0.805%	0.880%
0.05	50	0.569%	0.571%	0.530%	0.569%	0.607%	0.531%	0.569%	0.607%
0.05	75	0.465%	0.463%	0.440%	0.464%	0.490%	0.440%	0.464%	0.490%
0.05	100	0.403%	0.394%	0.384%	0.403%	0.421%	0.384%	0.403%	0.421%
0.05	150	0.329%	0.335%	0.316%	0.329%	0.341%	0.316%	0.329%	0.341%
0.05	200	0.285%	0.286%	0.275%	0.285%	0.294%	0.275%	0.285%	0.294%
0.10	10	1.831%	1.854%	1.454%	1.820%	2.224%	1.460%	1.825%	2.224%
0.10	25	1.158%	1.139%	1.009%	1.157%	1.311%	1.013%	1.158%	1.309%
0.10	50	0.819%	0.822%	0.742%	0.816%	0.895%	0.743%	0.817%	0.894%
0.10	75	0.669%	0.681%	0.618%	0.668%	0.720%	0.619%	0.668%	0.720%
0.10	100	0.579%	0.575%	0.541%	0.579%	0.617%	0.542%	0.579%	0.616%
0.10	150	0.473%	0.472%	0.447%	0.472%	0.499%	0.448%	0.473%	0.499%
0.10	200	0.409%	0.405%	0.391%	0.409%	0.428%	0.391%	0.409%	0.428%
0.15	10	2.284%	2.245%	1.709%	2.252%	2.840%	1.721%	2.256%	2.833%
0.15	25	1.444%	1.441%	1.216%	1.440%	1.678%	1.220%	1.441%	1.674%
0.15	50	1.021%	1.006%	0.906%	1.019%	1.137%	0.908%	1.020%	1.134%
0.15	75	0.834%	0.821%	0.758%	0.833%	0.910%	0.761%	0.833%	0.909%
0.15	100	0.722%	0.709%	0.666%	0.721%	0.778%	0.667%	0.722%	0.777%
0.15	150	0.590%	0.589%	0.551%	0.590%	0.627%	0.552%	0.590%	0.626%
0.15	200	0.511%	0.510%	0.481%	0.510%	0.539%	0.482%	0.510%	0.539%
0.20	10	2.686%	2.665%	1.896%	2.649%	3.431%	1.919%	2.666%	3.425%
0.20	25	1.699%	1.684%	1.384%	1.691%	2.004%	1.397%	1.695%	1.997%
0.20	50	1.201%	1.216%	1.043%	1.194%	1.357%	1.048%	1.195%	1.354%
0.20	75	0.981%	1.001%	0.875%	0.978%	1.089%	0.879%	0.980%	1.086%
0.20	100	0.849%	0.838%	0.772%	0.848%	0.927%	0.775%	0.849%	0.925%
0.20	150	0.693%	0.679%	0.640%	0.692%	0.742%	0.642%	0.692%	0.741%
0.20	200	0.601%	0.609%	0.560%	0.600%	0.640%	0.561%	0.600%	0.639%

Table 13: Simulation of LRADR standard deviation and different calculation approaches for $p = 10\%$