Analytic loss distributions of heterogeneous portfolios in the asset value credit risk model

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Abstract:

We provide an analytic solution to the asset value credit risk model that allows for heterogeneous correlations, default probabilities, recovery rates and exposures given certain regularity conditions are fulfilled.

Additionally, we extend the asset value model to include event risks such as country risk or dependencies between individual clients and derive analytic loss distributions and loss densities.

All results can be implemented in spreadsheet calculators such as Microsoft Excel or Lotus 1-2-3.

Keywords: credit portfolio risk, analytical loss distribution, country risk, microeconomic risk, asset value model

JEL Classification: C63, G21

Introduction

The need for analytic solutions to portfolio credit risk models that enable the risk manager to quickly assess the approximate risk of large portfolios has been felt since the beginning of modern portfolio risk management. Already in 1987 Oldrich Vasicek supplied an analytic solution to the classical asset value model for homogenous portfolios consisting of finitely many identical clients. In 1991 he extended the result to account for the asymptotic case of homogenous portfolios with infinitely many clients.

In 1997 Credit Suisse First Boston with Credit Risk+ published a new approach to analytical risk calculation based on the compound Poisson model known from insurance mathematics. Here the credit portfolio is understood as being composed of heterogeneous groups of exposures where, however, every group is homogenous in itself in the respect that it is assumed to contain a large number[1] of identical exposures.

More recently Schönbucher (2002) uses the mixing property of Archimedean copulas to obtain analytic loss distributions for homogenous portfolios in a model where asset returns have a multivariate Archimedean distribution such as a Gumbel or a Clayton distribution.

Wehrspohn (2003) derives analytic loss distributions for homogenous portfolios in a generalized version of the asset value model where the assumption of a normal distribution of asset returns is replaced by a general multivariate elliptic distribution. This is a direct extension of the classical asset value model in that for a given marginal distribution of asset returns the dependence structure of asset values is defined entirely by their linear correlation[2]. This generalization is not without consequence since it can be shown that there is no correlation model that finds less risk at high percentiles in a given portfolio than the classical asset value model[3].

In this article, we mainly stay within the classical asset value model and widen Vasiceks' original results in various respects. First, we allow for heterogeneous correlations, default probabilities, exposures, and losses given default between homogenous groups of clients. Second, we include country risk in the analysis of homogenous portfolios. Third, we allow for individual dependencies between clients in moderately heterogeneous portfolios where the

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1 Note that it is paramount for the Credit Risk+ model that each exposure has a large number of identical twins in the portfolio so that asymptotic results such as the Poisson approximation of the binomial distribution can be applied. If exposures are unique in the portfolio as is frequently the case for the usually very few largest exposures, portfolio risk is overestimated by the model. This error occurs because each exposure can default an infinite number of times in Credit Risk+. For details refer to Wehrspohn (2002), pp. 144ff.

2 We, therefore, also refer to the generalized asset value model as the generalized correlation model and to the classical asset value model as the normal correlation model.

3 See Wehrspohn (2003), theorem 6.

default of one client may cause the financial distress of another client and show the cascading effect caused by event risks. In all cases, we derive loss distributions and loss densities[4]. The proofs are given in the appendix.

1. Loss distributions of heterogeneous portfolios

The asset value model goes back to an article of Robert Merton (1974) and was later extended by KMV Corporation and Bhatia et al. (1997) to a credit portfolio model.

In Merton's model, all corporate debt is assumed to consist of a single zero bond. Consequently, it is assumed that the firm defaults if its asset value at the maturity of the zero bond is inferior to the face value of the bond. The asset value process is modeled as a geometric Brownian motion so that asset returns at maturity of the bond are normally distributed.

If a firm's default probability p is known, for instance by its rating, it can even be assumed without loss of generality that asset returns X are standard normally distributed, i.e. $X \sim N(0;1)$. In this model, a firm defaults if asset returns are inferior or equal to a default threshold α, i.e. if $X \leq \alpha$. Since this event occurs with probability p, we have $\alpha = \Phi^{-1}(p)$ where $\Phi^{-1}(\cdot)$ is the inverse cumulative standard normal distribution function.

To integrate dependencies between firms' default behavior into the model, a firm's asset value distribution is thought to be composed of two independent sources, a systematic factor[5] Y and an individual, idiosyncratic factor Z_i, where the factors are standard normally distributed and independent. Idiosyncratic factors of different firms i and j are equally stochastically independent. For firm i the asset returns can now be written as

$$X_i = a_i \cdot Y + \sqrt{1-a_i^2} \cdot Z_i$$

with weights $-1 \leq a_i \leq 1$. Note that the interpretation of the X_i's as asset returns is merely intuitive. It is irrelevant to know firms' true asset returns to solve the model. For this reason, we will rather refer to X_i as firm i's risk index.

The covariance of the risk indices of two clients i and j is

$$\mathrm{Cov}(X_i, X_j) = a_i \cdot a_j .$$

4 Various software tools that illustrate the results are available at www.wehrspohn.de and www.risk-and-evaluation.com.

5 Note that the systematic factor Y again can be modeled as being the convex combination of several systematic risk factors.

We assume in the following that all covariances are positive[6] and that $a_i \geq 0$ for all *i*.

With $\rho_i := a_i^2$ we can rewrite risk indices more intuitively as

$$X_i = \sqrt{\rho_i} \cdot Y + \sqrt{1-\rho_i} \cdot Z_i$$

because now ρ_i is the risk index correlation (or asset return correlation) of two identical firms.

As the frame for the derivation of loss distributions we define a moderately heterogeneous portfolio.

Definition:

A moderately heterogeneous portfolio H is the union of homogenous sub-portfolios H_j, $j = 1,...,h$,

$$H = \bigcup_{j=1}^{h} H_j$$

i.e. of sub-portfolios H_j that contain only identical clients with exposures e_j, risk index correlations ρ_j, probabilities of default p_j, expected loss given default rates[7] λ_j, and where all clients in the entire portfolio depend on the same unique systematic risk factor *Y*.

In other words, we define a moderately heterogeneous portfolio as the union of certain types of clients that commonly share the same systematic risk factor. Each type of client is represented by a homogenous sub-portfolio H_j containing n_j clients for $j = 1,...,h$.

Thus, in a moderately heterogeneous portfolio each firm's risk index X_i is given as

$$X_i = \sqrt{\rho_j} \cdot Y + \sqrt{1-\rho_j} \cdot Z_i$$

for a *j* and for $i = 1,...,N$ if $N := \sum_{j=1}^{h} n_j$ is the number of clients in portfolio *H*.

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6 We need that risk indices X_i are monotonously increasing (or decreasing) in the systematic factor *Y* for all $i = 1,..., N$. Equivalent results obtain if $a_j \leq 0$ for $j = 1,..., h$.

7 Note that we do not assume loss given default rates as being fixed. They may be random with the same mean (not necessarily the same distribution) being independent from all other random variables in the model such as systematic and idiosyncratic risk factors.

For ease of exposition, we assume in the following that $n_j = n$ for $j = 1,...,h$, and that E_j is the aggregated exposure in sub-portfolio *j* so that each firm in sub-portfolio *j* has exposure $e_j = E_j / n$.

We can now state the first result.

Theorem 1:

In a moderately heterogeneous portfolio with risk index correlations $0 \le \rho_j < 1$ for $j = 1,...,h$, the α-percentile of the asymptotic portfolio loss distribution is given as

$$\lim_{n\to\infty} L^{-1}(\alpha; p_1,..., p_h, e_1(n),..., e_h(n), \rho_1,..., \rho_h, \lambda_1,..., \lambda_h)$$

$$= L^{-1}(\alpha; p_1,..., p_h, E_1,..., E_h, \rho_1,..., \rho_h, \lambda_1,..., \lambda_h)$$

$$= \sum_{j=1}^{h} \lambda_j \cdot E_j \cdot \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot \Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_j}}\right)$$

with mean

$$\mu = \sum_{j=1}^{h} \lambda_j \cdot E_j \cdot p_j$$

and median

$$\text{Median} = \sum_{j=1}^{h} \lambda_j \cdot E_j \cdot \Phi\left(\frac{\Phi^{-1}(p_j)}{\sqrt{1-\rho_j}}\right).$$

Thus, in moderately heterogeneous portfolios the α-percentile of the loss distribution of the entire portfolio is just the sum of the α-percentiles of the loss distributions of the sub-portfolios. This also means that the portfolio value at risk, which is a percentile of the portfolio loss distribution for a specific value of α, can be calculated separately for all sub-portfolios and then be aggregated over the sub-portfolios by simple addition.

Moreover, the calculation of the percentile function only requires the solving of the normal cumulative distribution function, a feature that is provided by virtually all spreadsheet calculators such as Lotus 1-2-3 and Microsoft Excel.

Due to both of these properties, the component distributions

$$L_j^{-1}(\alpha) = \lambda_j \cdot E_j \cdot \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot \Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_j}} \right)$$

of the portfolio loss distribution in theorem 1 can particularly be used to calculate single exposures' (approximate) contributions to portfolio risk without having to perform a full-fleged portfolio analysis. The results can be employed, for instance, for risk limitation of exposures of single addresses or of certain portfolio segments.

To derive the loss density in moderately heterogeneous portfolios let

$$\mathbf{P}\{\text{Loss in}\, H \le x\} := L(x; p_1,\ldots,p_h, E_1,\ldots,E_h, \rho_1,\ldots,\rho_h, \lambda_1,\ldots,\lambda_h) = L(x)$$

be the cumulative distribution function (cdf) of the portfolio loss distribution.

This leads to

Theorem 2:

In a moderately heterogeneous portfolio with risk index correlations $0 < \rho_j < 1$ for $j = 1,\ldots,h$, the density of the asymptotic portfolio loss distribution is given as

$$L_d(x) = L_d(x; p_1,\ldots,p_h, E_1,\ldots,E_h, \rho_1,\ldots,\rho_h, \lambda_1,\ldots,\lambda_h)$$

$$:= \frac{d}{dx} L(x; p_1,\ldots,p_h, E_1,\ldots,E_h, \rho_1,\ldots,\rho_h, \lambda_1,\ldots,\lambda_h)$$

$$= \frac{\varphi\left(\Phi^{-1}(1-L(x))\right)}{\sum_{j=1}^{h} \frac{\lambda_j \cdot E_j \cdot \sqrt{\rho_j}}{\sqrt{1-\rho_j}} \cdot \varphi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot \Phi^{-1}(1-L(x))}{\sqrt{1-\rho_j}} \right)}$$

Since we have only stated the inverse cdf of the portfolio loss distribution in theorem 1 it is worth noting that the loss density can be calculated as

$$L_d(x) = L_d\left(L^{-1}(\alpha)\right) = \frac{\varphi\left(\Phi^{-1}(1-\alpha)\right)}{\sum_{j=1}^{h} \frac{\lambda_j \cdot E_j \cdot \sqrt{\rho_j}}{\sqrt{1-\rho_j}} \cdot \varphi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot \Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_j}} \right)}$$

for $x = L^{-1}(\alpha)$.

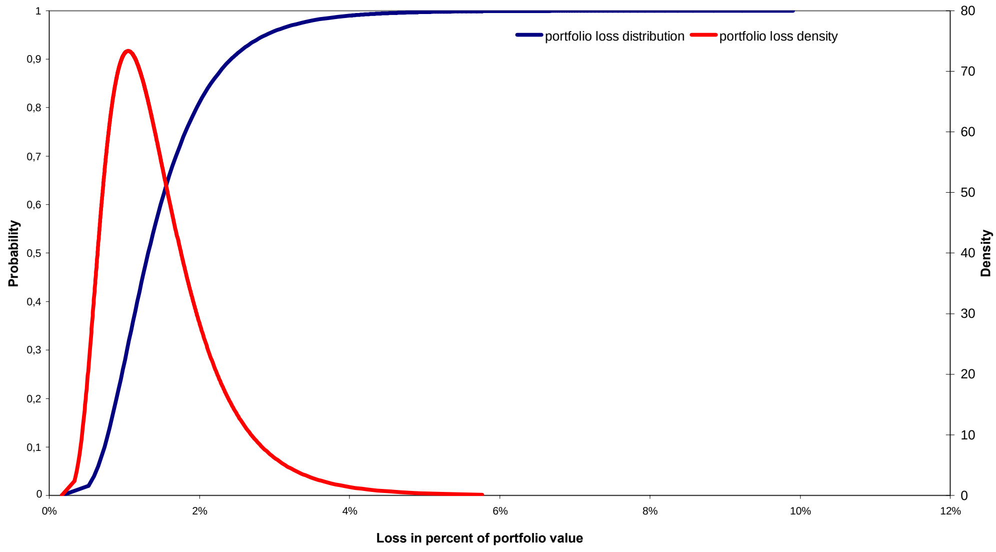

Figure 1: Portfolio loss distribution and portfolio loss density[8]

Figure 1 shows a typical loss distribution and loss density of a moderately heterogeneous portfolio. Note that the distribution is not necessarily unimodal. This feature is revealed if correlations are high.

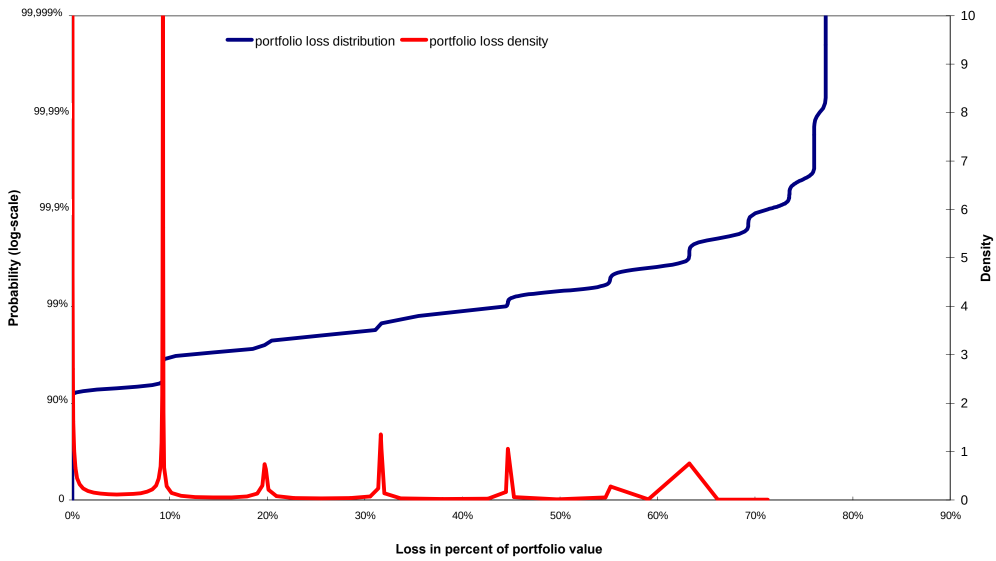

Figure 2: Multimodality of the loss distribution at high correlations[9]

Since firms within each group are almost perfectly correlated, the loss distribution is starting to be deformed into a step-function that only counts the defaults of entire groups weighted with their respective exposure. In turn the loss density degenerates. While still being continu-

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8 The portfolio underlying the exhibit consisted of ten sub-portfolio with the following characteristics

Group	I	II	III	IV	V	VI	VII	VIII	IX	X
Default Probability	0,01%	0,05%	0,1%	0,2%	0,4%	0,7%	1,2%	2,0%	3,0%	7,0%
Exposure	1	2	3	4	5	6	7	6	5	4
Loss Given Default	50%	55%	60%	65%	70%	75%	80%	85%	90%	100%
Correlation	20%	18%	16%	14%	12%	10%	8%	6%	4%	2%

9 The portfolio is the same as in footnote 8 with all correlations being set to 99.9%.

ous, it develops peaks at the left-edges of each step. In the limit for $\rho_j \to 1$, these peaks turn into point masses.

2. Loss distributions integrating country risk

A risk factor that is rarely considered in credit portfolio analysis is country risk[10]. For instance, creditors who are resident abroad can only serve their debts if international money transfers are not interrupted for reasons such as political or economic crises or the local central bank's lack of foreign currencies. The probability of the disruption of a country's international money transfers is typically measured by the countries rating as it is supplied by international rating agencies.

In portfolio analysis, country risk is usually thought to only influence the default probability of the clients who reside in a certain country. Consequently, it is tried to capture this effect by downgrading clients with a high individual creditworthiness to the rating of their home country.

However, this methodology neglects that country risk also establishes a dependence structure in a portfolio. If money transfers from a certain country are interrupted, this crisis does not only affect one client who resides there, but all creditors from that country simultaneously. As we will see below, this event risk effect is not appropriately modeled by a mere increase of individual default probabilities.

We incorporate country risk into the asset value model in the following way:

First, we assume that all ratings and default probabilities reflect the individual creditworthiness of each client unaffected of the rating of their country of origin. Second, we suppose that clients who are resident in a certain foreign country default either for reasons unrelated to the functioning of international money transfers or they all default simultaneously if their country defaults, independent of their individual situation.

To derive analytic loss distributions, consider a homogenous portfolio of clients with individual default probabilities p, risk index correlations ρ, loss given default rates λ, and a total portfolio exposure E. Assume that a fraction $s \in [0, 1]$ of the clients reside in one foreign country with default probability p_c.

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[10] Exceptions are Credit Portfolio View (see Wilson (1997a,b)) and the Credit Risk Evaluation model (see Wehrspohn (2002)).

Theorem 3:

In the asset value model including country risk, for risk index correlations $0 < \rho \le 1$ the loss distribution (cdf) of a homogenous portfolio is given by

$$\mathbf{P}\{\text{Loss} \le x\} = L(x; p, \rho, \lambda, E, s, p_c)$$

$$= \begin{cases} (1-p_c)\cdot\Phi\left(\dfrac{\sqrt{1-\rho}\cdot\Phi^{-1}(x/(\lambda\cdot E)) - \Phi^{-1}(p)}{\sqrt{\rho}}\right) + p_c\cdot A(x) & \text{if } x \le \lambda\cdot E \\ 1 & \text{else} \end{cases}$$

with

$$A(x) = \begin{cases} 0 & \text{if } x \le s\cdot\lambda\cdot E \\ \Phi\left(\dfrac{\sqrt{1-\rho}\cdot\Phi^{-1}\left(\dfrac{x/(\lambda\cdot E) - s}{1-s}\right) - \Phi^{-1}(p)}{\sqrt{\rho}}\right) & \text{else} \end{cases}$$

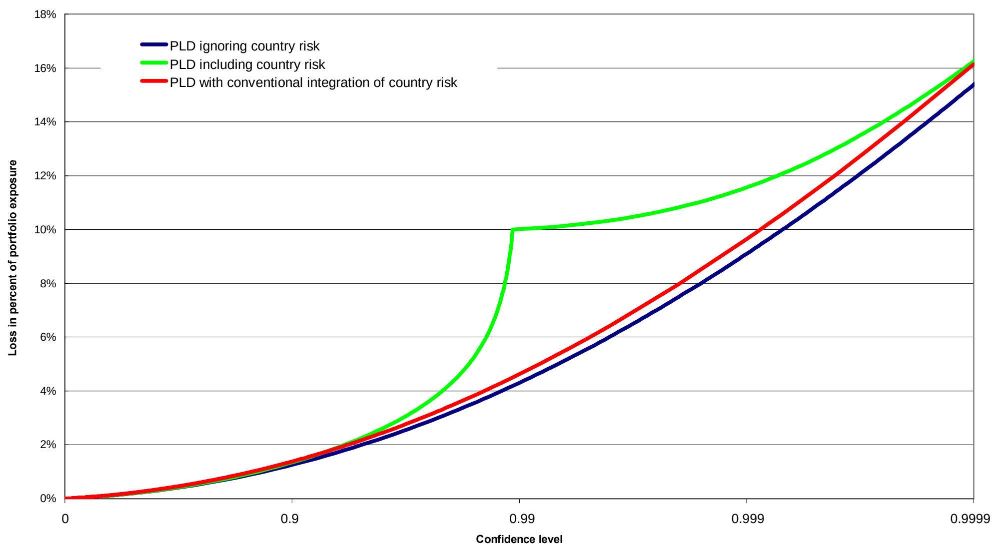

Figure 3: Country risk effect[11]

[11] The portfolio underlying the figure has the characteristics $p = 0.5\%$, $\rho = 20\%$, $\lambda = 100\%$, $s = 10\%$, and $p_C = 1\%$.

Figure 3 illustrates the effect of the inclusion of country risk into portfolio risk calculations. At low percentiles, the consideration of country risk does not make a great difference. Shortly before a confidence level of $1-p_c$, however, portfolio losses sharply rise to take account of the consequences of the default of the foreign country. For higher percentiles, the discrepancy between the loss distribution that correctly includes and the loss distribution that ignores country risk slowly vanishes again.

Note that the conventional method to integrate country risk in portfolio risk analysis slightly overestimates the country risk effect at low, but considerably underestimates it at high confidence levels. This is owing to the fact that the conventional method understates the foreign clients' overall default probability and, more importantly, does not capture the remarkable shift in the dependence structure within the portfolio caused by country risk.

We formulate the loss density that includes country risk as

Theorem 4:

In the asset value model including country risk, for risk index correlations $0<\rho<1$ the loss density of a homogenous portfolio is given by

$$L_d(x; p, \rho, \lambda, E, s, p_c)$$

$$= \begin{cases} (1-p_c)\cdot\frac{\sqrt{1-\rho}}{\lambda\cdot E\cdot\sqrt{\rho}}\varphi\left(\frac{\Phi^{-1}(p)-\sqrt{1-\rho}\cdot\Phi^{-1}(x/(\lambda\cdot E))}{\sqrt{\rho}}\right)\frac{1}{\varphi(\Phi^{-1}(x/(\lambda\cdot E)))}+p_c\cdot B(x) & \text{if } x\le\lambda\cdot E \\ 0 & \text{else} \end{cases}$$

with

$$B(x)=\begin{cases} 0 & \text{if } x\le s\cdot\lambda\cdot E \\ \frac{\sqrt{1-\rho}}{\lambda\cdot E\cdot(1-s)\cdot\sqrt{\rho}}\varphi\left(\frac{\Phi^{-1}(p)-\sqrt{1-\rho}\cdot\Phi^{-1}\left(\frac{x/(\lambda\cdot E)-s}{1-s}\right)}{\sqrt{\rho}}\right)\cdot\frac{1}{\varphi\left(\Phi^{-1}\left(\frac{x/(\lambda\cdot E)-s}{1-s}\right)\right)} & \text{else} \end{cases}$$

The theorem can be proved by derivation of the loss distribution.

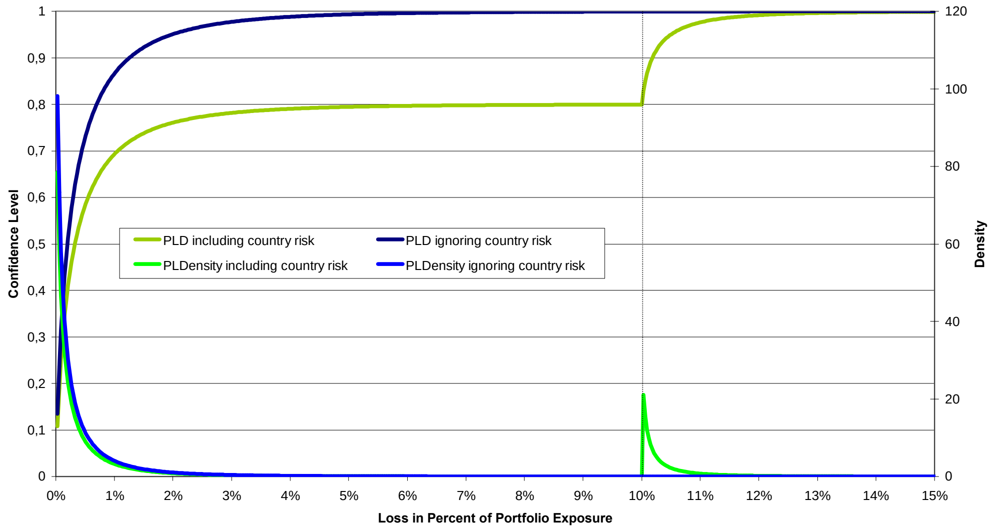

Figure 4: Country risk effect on portfolio loss densities[12]

Figure 4 gives an example of the country risk effect on portfolio loss densities. It can be clearly seen that the loss distribution is bimodal even in case of homogenous portfolios with a second mode at portfolio losses slightly higher than the share of foreign exposure in the portfolio.

3. Loss distributions integrating individual dependencies between clients

One of the most prominent reasons of default of small an medium sized companies throughout the world is the financial distress of a major business partner. This is another risk factor that is usually neglected in credit portfolio analysis.

We include this new dependence structure into the asset value model by assuming that each client who is individually related to another client defaults with a certain 'contamination probability' if the client he is dependent on has defaulted (unless he already has defaulted for other reasons before). I.e., we suppose that the default of his very important business partner draws him into a crisis that leads to a bad end with a certain probability. However, it is also possible that he recovers and does not default himself.

In order to be able to derive analytic loss distributions, we again suppose a moderately heterogeneous portfolio where within a fraction $\nu \in [0,1]$ of all clients each is related to exactly one other client and that there is no client upon whom two or more others depend. The con-

[12] The underlying portfolio has the characteristics $p = 0.5\%$, $\rho = 5\%$, $\lambda = 100\%$, $s = 10\%$, and $p_C = 20\%$.

tamination probability π that a client is drawn into insolvency if his partner defaults is assumed to be the same for the entire portfolio.

In this situation portfolio risk develops in cascades. In the starting round, clients default spontaneously, depending only on the systematic risk factor *Y*. In the next round, additional clients who survived the starting round default in reaction to the collapse of their trade partners. In the second round, the victims of the victims are added to the loss score and so on. This effect is summed up in

Theorem 5:

In the asset value model including individual dependencies, for risk index correlations $0 \leq \rho_j < 1$ for $j = 1,\ldots,h$, the α-percentile of the asymptotic portfolio loss distribution of a moderately heterogeneous portfolio after *n* rounds of successive defaults is given by

$$L_n^{-1}(\alpha) = a_0(\alpha) + \nu \cdot \sum_{i=1}^{n} a_i(\alpha)$$

where $a_i(\alpha)$ is defined recursively for $i = 1,\ldots,n$ as

$$a_i(\alpha) := \pi \cdot a_{i-1}(\alpha) \cdot \left(1 - \sum_{k=0}^{i-1} a_k(\alpha)\right)$$

and

$$a_0(\alpha) := \sum_{j=1}^{h} \lambda_j \cdot E_j \cdot \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot \Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_j}}\right).$$

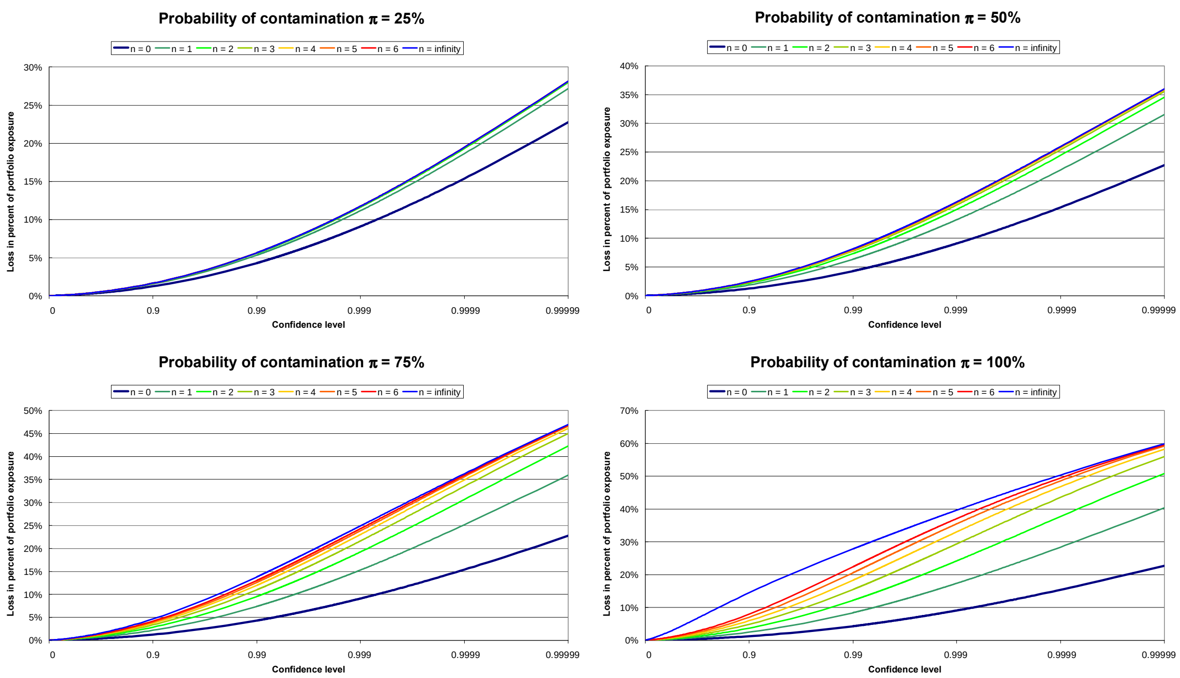

Figure 5: Cascading effect of microeconomic dependencies[13]

Figure 5 illustrates theorem 5 for various contamination probabilities. Note that even for $\pi = 100\%$ the portfolio loss distribution converges to a non-degenerate limit distribution for $n \to \infty$. However, the cascading effect of individual dependencies is considerable already for low values of π and n indicating that risk managers should well be aware of this type of portfolio dependence.

We now derive the portfolio loss density

Theorem 6:

In the asset value model including individual dependencies, for risk index correlations $0 < \rho_j < 1$ for $j = 1,...,h$, the density of the asymptotic portfolio loss distribution of a moderately heterogeneous portfolio after n rounds of successive defaults is given by

$$L_{n,d}(x) = \frac{1}{a_0'(L_n(x)) + \nu \cdot \sum_{i=1}^{n} a_i'(L_n(x))}$$

with

$$a_i'(L_n(x)) = \pi \cdot a_{i-1}'(L_n(x)) \cdot \left(1 - \sum_{k=0}^{i-1} a_k(L_n(x))\right) - \pi \cdot a_{i-1}(L_n(x)) \cdot \sum_{k=0}^{i-1} a_k'(L_n(x))$$

for $i = 1,...,n$ and

[13] With $h = 1$ and $\nu = 1$ clients are assumed to have a default probability of 0.5%, a loss given default rate of 100%, and risk index correlations of 20%.

$$a_0'(L_n(x)) = \sum_{j=1}^{h} \frac{\lambda_j \cdot E_j \cdot \sqrt{\rho_j}}{\sqrt{1-\rho_j}} \cdot \varphi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot \Phi^{-1}(1 - L_n(x))}{\sqrt{1-\rho_j}} \right) \cdot \frac{1}{\varphi(\Phi^{-1}(1 - L_n(x)))}$$

The theorem can be proved along the same lines as theorem 2.

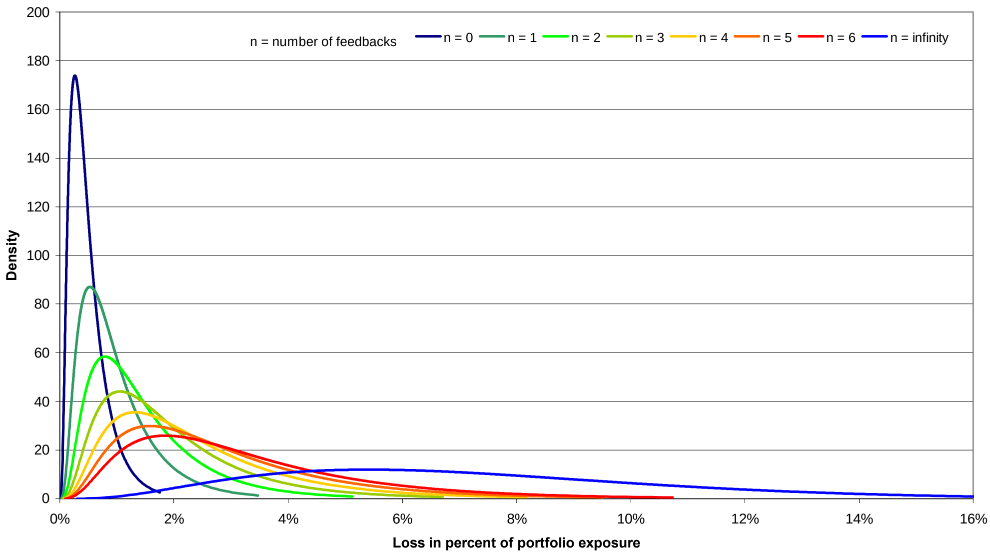

Figure 6: Cascading portfolio loss densities resulting from microeconomic risk[14]

Figure 6 shows portfolio loss densities for various numbers of feedbacks. As is to be expected, loss densities move to the right and become more and more flat with increasing value of n.

Conclusion

Extending a classical result by Vasicek, we have derived analytical loss distributions and loss densities for moderately heterogeneous portfolios that allow for heterogeneous correlations, default probabilities, loss given default rates, and exposures.

Moreover, we have generalized the asset value credit risk model to include country risk and microeconomic dependencies between clients, respectively, two structural properties that occur frequently in credit portfolios, but are rarely taken account of in credit portfolio risk modeling. Again we have derived analytical loss distributions and loss densities under specific regularity conditions.

All formulae can be used by risk managers to approximately assess the risk of large portfolios without the need of time-consuming complex computer simulations.

[14] With $h = 1$ and $\nu = 1$, clients are assumed to have a default probability of 0.5%, a loss given default rate of 100%, and risk index correlations of 5%.

Appendix:

Proof of theorem 1:

By construction of the asset value model, client *i* in sub-portfolio H_j defaults if

$$\begin{aligned} X_i &= \sqrt{\rho_j} \cdot Y + \sqrt{1-\rho_j} \cdot Z_i \le \Phi^{-1}(p_j) \\ \Leftrightarrow Z_i &\le \frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot Y}{\sqrt{1-\rho_j}} \end{aligned}$$

for $i = 1,...,n$ and $j = 1,...,h$.

Hence, client *i*'s probability of default conditional to *Y* is given as

$$\mathbf{P}\{\text{client } i \text{ defaults} \mid Y\} = \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot Y}{\sqrt{1-\rho_j}} \right)$$

because Z_i is standard normally distributed for $i = 1,...,n$.

Moreover, since the idiosyncratic components Z_i of clients' risk indices are stochastically independent, it follows from the law of large numbers that the percentage of clients defaulting in each sub-portfolio given *Y* is almost surely equal to their conditional probability of default if $n \to \infty$.

Asymptotically the number of defaulting clients in each sub-portfolio goes to infinity as well, if the conditional default probability is positive. Thus, again by the law of large numbers, the loss conditional to *Y* in sub-portfolio H_j is equal to

$$\text{Loss in } H_j \mid Y = \lambda_j \cdot E_j \cdot \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot Y}{\sqrt{1-\rho_j}} \right)$$

since the individual loss given default rates in H_j are stochastically independent and limited with the same mean λ_j for $j = 1,...,h$.

The loss in the entire portfolio *H* conditional to *Y* is then given as

$$\text{Loss in } H \mid Y = \sum_{j=1}^{\mathrm{h}} (\text{Loss in } H_j \mid Y) = \sum_{j=1}^{h} \lambda_j \cdot E_j \cdot \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot Y}{\sqrt{1-\rho_j}} \right)$$

because all clients in *H* depend exclusively on the same systematic risk factor *Y*.

The inverse cumulative loss distribution function can immediately be derived from this expression for portfolio losses because the systematic risk factor Y is the only remaining random component in portfolio losses and the conditional portfolio loss given Y is monotonously decreasing in Y. The α-percentile of the portfolio loss distribution, therefore, maps one-to-one to the $(1-\alpha)$-percentile of the distribution of Y. The $(1-\alpha)$-percentile of Y is given by $\Phi^{-1}(1-\alpha)$ because Y is standard normally distributed.

The formula for the mean results because firms in sub-portfolio j have default probability p_j. The median is obtained for $\alpha = 1/2$.

□

Proof of theorem 2:

The derivative of the inverse cdf with respect to α is

$$L_d^{-1}(\alpha) = \frac{d}{d\alpha} L^{-1}(\alpha) = \sum_{j=1}^{h} \frac{\lambda_j \cdot E_j \cdot \sqrt{\rho_j}}{\sqrt{1-\rho_j}} \cdot \varphi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho_j} \cdot \Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_j}} \right) \cdot \frac{1}{\varphi(\Phi^{-1}(1-\alpha))}$$

The theorem is then implied by

$$L_d(x) = \frac{1}{L_d^{-1}(L(x))}.$$

□

Proof of theorem 3:

We go through the components of the loss distribution separately.

First of all, the maximum possible loss of the portfolio is $\lambda \cdot E$ by construction of the model implying that $\mathbf{P}\{\text{Loss} \le x\} = 1$ if $x \ge \lambda \cdot E$.

Let $x < \lambda \cdot E$. By definition of the conditional probability, it is

$$\mathbf{P}\{\text{Loss} \le x\} = (1 - p_c) \cdot \mathbf{P}\{\text{Loss} \le x \mid \text{no country default}\} + p_c \cdot \mathbf{P}\{\text{Loss} \le x \mid \text{country default}\}$$

Setting $h = 1$ in theorem 1 and solving of the inverse cdf for x, it can be shown that

$$\mathbf{P}\{\text{Loss} \le x \mid \text{no country default}\} = \Phi\left(\frac{\sqrt{1-\rho} \cdot \Phi^{-1}(x/(\lambda \cdot E)) - \Phi^{-1}(p)}{\sqrt{\rho}} \right)$$

In case of default of the country, all foreign clients simultaneously cannot meet their due payments, thus

$$\mathbf{P}\{\text{Loss} < s\cdot\lambda\cdot E \mid \text{country default}\} = 0\,.$$

Losses $x > s\cdot\lambda\cdot E$ can here only result from the insolvency of some of the domestic clients who are unaffected by the foreign country's default. Taking loss given default rates and portfolio exposure into consideration, exactly the fraction $\frac{x/(\lambda\cdot E)-s}{1-s}$ of domestic clients has to default to ensure excess losses to equal $x - s\cdot\lambda\cdot E$, thus

$$\mathbf{P}\{\text{Loss} \le x \mid x > s\cdot\lambda\cdot E \text{ and country default}\} = \Phi\left(\frac{\sqrt{1-\rho}\cdot\Phi^{-1}\left(\frac{x/(\lambda\cdot E)-s}{1-s}\right)-\Phi^{-1}(p)}{\sqrt{\rho}}\right)$$

□

Proof of theorem 5:

$a_0(\alpha)$ results immediately from theorem 1.

For $\nu = 1$, due to the infinite number of clients in the asymptotic portfolio it follows from the law of large numbers that the additional fraction of clients who default in round i is equal to its expected value, i.e. $\pi\cdot a_{i-1}(\alpha)$. However, with probability 1 the fraction $\sum_{k=1}^{i-1} a_k(\alpha)$ of these has already defaulted before for other reasons. Thus, we have

$$a_i(\alpha) := \pi\cdot a_{i-1}(\alpha)\cdot\left(1-\sum_{k=0}^{i-1} a_k(\alpha)\right)$$

almost surely. $L_n^{-1}(\alpha)$ then is just the sum of all losses in rounds 1 to n:

$$L_n^{-1}(\alpha) = a_0(\alpha) + \sum_{i=1}^{n} a_i(\alpha)$$

For $\nu < 1$ additional losses resulting from individual dependencies only stem from the respective fraction of all clients, thus, we have

$$L_n^{-1}(\alpha) = a_0(\alpha) + \nu\cdot\sum_{i=1}^{n} a_i(\alpha)$$

□

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