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The illusions of dynamic replication

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1. Introduction

How well does options pricing theory really work, and how dependent is it on the notion of dynamic replication? In this note we describe what many practitioners know from long and practical experience: (i) dynamic replication doesn't work as well as students are taught to believe; (ii) most derivatives traders rely on it as little as possible; and (iii) there is a much simpler way to derive many option pricing formulas: many of the results of dynamic option replication can be obtained more simply, by regarding (as many practitioners do) an options valuation model as an interpolating formula for a hybrid security that correctly matches the boundary values of the ingredient securities that constitute the hybrid.

2. Replication

The logic of replication is that a security whose payoff can be replicated purely by the continuous trading of a portfolio of underlying securities is *redundant*; its value can be derived from the value of the underlying replicating portfolio, requiring no utility function or risk premium applied to expected values. The fair value of the replicated security follows purely from riskless arbitrage arguments.

The method of static replication for valuing securities was well known, but prior to Black and Scholes (1973) the possibility of dynamic replication was unexplored, although there had been hints of the approach, as in Arrow (1953). What distinguishes the Black–Scholes–Merton model is the dynamic replication of the portfolio and the economic consequences of this argument, rather than, as is frequently asserted in the literature, the option pricing equation *per se*.

We shall show that the Black–Scholes option pricing formula could have been derived much earlier by requiring that a portfolio consisting of a long position in a call and a short position in a put, valued by the traditional discounted expected value of their payoffs, must statically replicate a forward contract.

3. Arguments for skepticism

There are a variety of empirical arguments that justify some skepticism about the efficacy of dynamic hedging as a framework for options valuation.

- Options are currently priced and traded on myriads of instruments—live commodities, agricultural products, perishable goods, and extremely illiquid equity securities—where dynamic replication cannot possibly be achieved. Yet these options are priced with the *same* models and software packages as are options on those rare securities where dynamic replication is feasible.
- Even where dynamic replication is feasible, the theory requires continuous trading, a constraint that is unachievable in practice. The errors resulting from discrete hedging, as well as the transaction costs involved, are prohibitive, a point that has been investigated extensively in the literature (see, for example, Taleb (1997, 1998)).
- In addition, market-makers, who are in the business of manufacturing long and short option positions for their clients, do not hedge every option dynamically; instead they hedge only their extremely small *net* position. Thus, the effect of the difference between dynamic and static hedging on their portfolio is extremely small.
- Dynamic replication assumes continuous asset price movements, but real asset prices can move discontinuously, destroying the possibility of accurate replication and providing a meaningful likelihood of bankruptcy for any uncovered option seller who does not have *unlimited* capital.

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- All manner of exotic and even hybrid multidimensional derivative structures have proliferated in the past decade, instruments of such complexity that dynamic replication is clearly practically impossible. Yet they are priced using extensions of standard options models.

Hakansson's so-called paradox (Hakansson 1979, Merton 1992) encapsulates the skepticism about dynamic replication: if options can only be priced because they can be replicated, then, since they can be replicated, why are they needed at all?

4. The logic of dynamic hedging

Let us review the assumptions about dynamic replication that lead to the Black–Scholes equation for European options on a single stock.

In the Black–Scholes picture a stock S is a primitive security, primitive in the sense that its payoff cannot be replicated by means of some other security. An option C whose payoff depends through a specified payoff function of S at some expiration time T is a derivative security.

Assume that the underlying stock price S undergoes geometric Brownian motion with expected return μ and return volatility σ. A short position in the option C with price $C(S,t)$ at time t can be hedged by purchasing $\partial C/\partial S$ shares of stock against it.

The hedged portfolio $\Pi = -C + \partial C/\partial S$ S consisting of a short position in the option and a long position in Δ shares of the underlying stock will have no instantaneous linear exposure to the stock price S.

Note that the immediate effect of this hedge is to remove all immediate dependence of the value of portfolio Π on the expected return μ of the stock.

$$E[\Delta\Pi] = -\partial C/\partial S E[\Delta S] - \frac{1}{2}\partial^2 C/\partial S^2 E[\Delta S^2] - \partial C/\partial t \Delta t + \partial C/\partial S E[\Delta S].$$

We† can see how the first and last terms cancel each other, eliminating $E[\Delta S]$ from the expectation of the variations in the hedged portfolio.

The portfolio of option and stock has not yet becomes a *riskless* instrument whose return is determined. We need another element, the stream of subsequent dynamic hedges.

With continuous rehedging, the instantaneous profit on the portfolio per unit time is given by

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t},$$

assuming for simplicity that the riskless interest rate is zero.

If the future return volatility σ of the stock is known, this profit is deterministic and riskless. If there is to be no arbitrage on any riskless position, then the instantaneous profit must be zero, leading to the canonical Black–Scholes equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} = 0, \tag{1}$$

which can be solved for boundary conditions corresponding to a simple European call to yield the Black–Scholes formula.

Note that the Nobel committee upon granting the Bank of Sweden Prize in honour of Alfred Nobel, provided the following citation: 'Black, Merton and Scholes made a vital contribution by showing that it is in fact not necessary to use any risk premium when valuing an option. This does not mean that the risk premium disappears; instead it is already included in the stock price.'‡ It is for having removed the effect of μ on the value of the option, *and not for rendering the option a deterministic and riskless security*, that their work is cited.

The effect of the subsequent stream of secondary dynamic hedges is to render the option riskless, not, as it is often assumed, to remove the risk of the exposure to the underlying security. The more we hedge, the more the option becomes (under the Black–Scholes assumptions) a deterministic payoff—but, again, under a set of very precise and idealized assumptions, as we will see next.

5. Dynamic hedging and its discontents

The Black–Scholes–Merton formalism relies upon the following central assumptions:

(1) constant (and known) σ;
(2) constant and known carry rates;
(3) no transaction costs;
(4) frictionless (and continuous) markets.

Actual markets violate all of these assumptions.

- Most strikingly, the implied volatility smile is incompatible with the Black–Scholes–Merton model, which leads to a flat implied volatility surface. Since the option price is incompatible with the Black–Scholes formula, the correct hedge ratio is unknown.
- One cannot hedge continuously. Discrete hedging causes the portfolio Π to become risky before the next rebalancing. One can think of this as a sampling error of order $1/(\sqrt{2N})$ in the stock's volatility, where N is the number of rebalancings. Hedge 50 times on a three-month option rather than continuously, and the standard deviation of the error in the replicated option price is about 10%, a significant mismatch.

†We are taking the equality in expectation because we are operating in discrete time not at the limit of Δt going to 0.
‡See www.Nobel.se

- In addition to the impossibility of continuous hedging, transaction costs at each discrete rehedging impose a cost that make an options position worth less than the Black–Scholes value.
- Future carry rates are neither constant nor known.
- Furthermore, future volatility is neither constant nor known.
- More radically, asset price distributions have fat tails and are inadequately described by the geometric Brownian motion assumed by Markowitz's mean-variance theory, the Capital Asset Pricing Model and options theory itself.

Furthermore, practitioners know from bitter experience that dynamic replication is a much more fragile procedure than static replication: a trading desk must deal with transactions costs, liquidity constraints, the need for choosing price evolution models and the uncertainties that ensue, the confounding effect of discontinuous asset price moves, and, last but by no means least, the necessity for position and risk management software.

6. Options valuation by expectations and static replication

Practitioners in derivatives markets tend to regard options models as interpolating formulas for hybrid securities. A convertible bond, for example, is part stock, part bond: it becomes indistinguishable from the underlying stock when the stock price is sufficiently high, and equivalent to a corporate bond when the stock price is sufficiently low. A convertible bond valuation model provides a formula for smoothly interpolating between these two extremes. In order to provide the correct limits at the extremes, the model must be calibrated by static replication. A convertible model that doesn't replicate a simple corporate bond at asymptotically low stock prices is fatally suspect.

One can view the Black–Scholes formula in a similar light. Assume that a stock S that pays no dividends has future returns that are lognormal with volatility σ. A plausible and time-honoured *actuarial* way to estimate the value at time t of a European call C with strike K expiring at time T is to calculate its expected discounted value, which is given by

$$\begin{aligned} C(S,t) &= e^{-r(T-t)}(E[S-K]_+) \\ &= e^{-r(T-t)}\Big\{S\,e^{\mu(T-t)}N(d_1) - KN(d_2)\Big\}, \end{aligned} \quad (2)$$

where r is the appropriate but unknown discount rate, still unspecified and μ is the unknown expected growth rate for the stock.

The analogous actuarial formula for a put P is given by

$$\begin{aligned} P(S,t) &= e^{-r(T-t)}(E[K-S]_+) \\ &= e^{-r(T-t)}\Big\{KN(-d_2) - Se^{\mu(T-t)}N(-d_1)\Big\}, \end{aligned} \quad (3)$$

where

$$d_{1,2} = \frac{\ln[S\,e^{\mu(T-t)}/K] \pm [\sigma^2(T-t)/2]}{\sigma\sqrt{T-t}}. \quad (4)$$

A dealer or market-maker in options, however, has additional consistency constraints. As a manufacturer rather than a consumer of options, the market-maker must stay consistent with the value of his raw supplies. He must notice that a portfolio $F = C - P$ consisting of a long position in a call and a short position in a put with the same strike K has exactly the same payoff as a forward contract with expiration time T and delivery price K whose fair current value is

$$F = S - K\,e^{-R(T-t)}, \quad (5)$$

where R is the zero-coupon riskless discount rate for the time to expiration.

The individual formulas of equations (2) and (3) must be calibrated to be consistent with equation (5). If they are not, the market-maker will be valuing his options, stock and forward contracts inconsistently, despite their underlying similarity. What conditions are necessary to satisfy this?

Combining equations (2) and (3) we obtain

$$F = C - P = e^{-r(T-t)}\Big\{S\,e^{\mu(T-t)} - K\Big\}. \quad (6)$$

The requirement that equations (5) and (6) be consistent dictates that both the appropriate discount rate r and the expected growth rate μ for the stock in the options formula be the zero-coupon discount rate R. These choices make equation (2) equivalent to the Black–Scholes formula.

A similar consistency argument can be used to derive the values of more complex derivatives, dependent on a larger number of underlyers, by requiring consistency with the values of all tradable forwards contracts on those underlyers. For an application of this method to valuing quanto options, see Derman *et al.* (1998).

7. From Bachelier to Keynes

Let us zoom back into the past. Assume that in 1973, there were puts and calls trading in the market-place. The simple put–call parity argument would have revealed that these can be combined to create a forward contract.

John Maynard Keynes was the first to show that the forward need not be priced by the expected return on the stock, the equivalent of the μ we discussed earlier, but by the arbitrage differential, namely, the equivalent of $r - d$. This follows the exposition of the formula that was familiar to every institutional foreign exchange trader.

> If by lending dollars in New York for one month the lender could earn interest at the rate of $5\frac{1}{2}$% per annum, whereas by lending sterling in London for

> one month he could only earn interest at the rate of 4%, then the preference observed above for holding funds in New York rather than in London is wholly explained. That is to say, forward quotations for the purchase of the currency of the dearer money market tend to be cheaper than spot quotations by a percentage per month equal to the excess of the interest which can be earned in a month in the dearer market over what can be earned in the cheaper. Keynes (1923, 2000)

Between Bachelier and Black–Scholes, there were several researchers who produced formulas similar to that of Black–Scholes, differing from it only by their use of a discount rate that was not riskless. While Bachelier had the Black–Scholes equation with no drift and under an arithmetic Brownian motion, others added the drift, albeit a nonarbitrage derived one, in addition to the geometric motion for the dynamics. Of these equations we can cite Sprenkle (1961), Boness (1964), Samuelson (1965), and Samuelson and Merton (1969). All of their resultant pricing equations involved unknown risk premiums that would have been determined to be zero had they used the put–call replication argument we illustrated above. Furthermore, the put–call parity constraint was already present in the literature (see Stoll, 1969).

8. Conclusion

Dynamic hedging is neither strictly required nor strictly necessary for plausibly valuing options; it is less relied upon in practice than is commonly believed. Much of financial valuation does not require such complexity of exposition, elegant though it may be†. The formulas it leads to can often be obtained much more simply and intuitively by constrained interpolation. Finally, the pricing of contingent claims by interpolation and static replication opens the door to valuing options on assets without necessarily demanding that such assets have finite square variation, and thus sets the grounds for the use of a richer class of distributions with finite first moment.

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References

Arrow, K., The role of securities in the optimal allocation of risk-bearing. *Econometrie*, 1953.

Black, F. and Scholes, M., The pricing of options and corporate liabilities. *J. Polit. Econ.*, 1973, **81**, 637–654.

Boness, A.J., Elements of a theory of stock option value. *J. Polit. Econ.*, 1964, **81**(3), 637–654.

Derman, E., Karasinski, P. and Wecker, J., Understanding guaranteed-exchange-rate options. In *Currency Derivatives*, edited by D. DeRosa, 1998 (Wiley: New York).

Hakkanson, N.H., The fantastic world of finance: progress and the free lunch. *J. Financ. Quant. Anal.*, 1979, **14**, 717–734.

Keynes, J.M., *A Tract on Monetary Reform*, 1923 (2000) (Prometheus Books: Amherst).

MacKenzie, D., An equation and its worlds: bricolage, exemplars, disunity and performativity in financial economics. *Soc. Stud. Sci.*, 2003, **33**(6), 831–868.

Merton, R.C., *Continuous Time Finance*, 1992 (Blackwell: London).

Samuelson, P.A., Rational theory of warrant pricing. *Ind. Manag. Rev.*, 1965, **6**(2), 13–32.

Samuelson, P.A. and Merton, R.C., A complete model of asset prices that maximizes utility. *Ind. Manag. Rev.*, 1969, **10**, 17–46.

Sprenkle, C.M., Warrant prices as indicators of expectations and preferences. *Yale Economic Essays*, 1961, **1**(2), 178–231 [reprinted in *The Random Character of Stock Market Prices*, edited by P.H. Cootner, 1967 (MIT Press: Cambridge, MA)].

Stoll, H.R., The relationship between put and call prices. *J. Finance*, 1969, **24**(5), 801–824.

Taleb, N.N., *Dynamic Hedging: Managing Vanilla and Exotic Options*, 1997 (Wiley: New York).

Taleb, N.N., Replication d'options et structure de marché, Report, Université Paris IX Dauphine, 1998.

†For a sociological study of the Black and Scholes formula, see MacKenzie (2003).