Simulations with exact means and covariances

Attilio Meucci presents a simple method to generate scenarios from multivariate elliptical distributions with given sample means and covariances, and shows an application to the risk management of a book of options.

To perform risk and portfolio management, we must represent the distribution of the risk factors that affect the market. The most flexible approach is in terms of scenarios and their probabilities, which includes historical scenarios, pure Monte Carlo and importance sampling (see Glasserman, 2004).

Here, we present a simple method to generate scenarios from elliptical distributions with given sample means and covariances. This is very important in applications such as mean-variance portfolio optimisation, which are heavily affected by incorrect representations of the first two moments.

This quickly becomes intractable for large Monte Carlo simulations. Instead, our method is a multivariate generalisation of the intuitive shift/rescaling that appears in, for example, Boyle, Broadie & Glasserman (1995) and they satisfy the zero-mean condition

\[ \hat{x}_j = x_j - \mu \], \( j = 1, \ldots, J \]

Next, we apply a linear transformation to the scenarios \( \mathbf{y}_j \), which again preserves normality:

\[ \mathbf{y}_j = \mathbf{B} \mathbf{x}_j, \quad j = 1, \ldots, J \]

For any choice of the invertible matrix \( \mathbf{B} \), the sample mean is null: \( \hat{\mathbf{m}} = \mathbf{0} \). To determine \( \mathbf{B} \), we impose that the sample covariance \( \hat{\mathbf{S}} \) matches the desired covariance \( \mathbf{S} \). Using the affine equivariance of the sample covariance (see, for example, Meucci, 2005), we obtain a matrix Riccati equation:

\[ \mathbf{S} = \mathbf{B} \hat{\mathbf{S}} \mathbf{B}^T, \quad \mathbf{B} = \mathbf{B}' \]

To solve this equation we follow Petkov, Christov & Konstantinov (1991). First, we define the Hamiltonian matrix:

\[ \mathbf{H} = \begin{pmatrix} 0 & -\hat{\mathbf{S}} \\ -\mathbf{S} & 0 \end{pmatrix} \]

Consider a multivariate normal distribution:

\[ \mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{S}) \]  

where \( \mathbf{m} \) is an arbitrary expected value and \( \mathbf{S} \) is an arbitrary covariance matrix. Consider the representation of this distribution in terms of the probability-scenario pairs \( (p_j, \mathbf{x}_j) \), \( j = 1, \ldots, J \). Our aim is to ensure that:

\[ \hat{\mathbf{m}}_j = \mathbf{m}, \quad \hat{\mathbf{S}}_j = \mathbf{S} \]

where:

\[ \mathbf{m} = \sum_{j=1}^{J} p_j \mathbf{x}_j, \quad \hat{\mathbf{m}}_j = \sum_{j=1}^{J} p_j \mathbf{x}_j - \mathbf{m}, \hat{\mathbf{m}}_j \]

\[ \hat{\mathbf{S}}_j = \mathbf{S} \]

denote the sample mean and sample covariance of \( (p_j, \mathbf{x}_j) \). To do so, one can either constrain the probabilities \( p_j \) or the scenarios \( \mathbf{x}_j \). The former approach, pursued in, for example, Avellaneda (1999), D’Amico, Fusai & Tagliani (2003), Glasserman & Yu (2005) and Meucci (2008), is very flexible, but for large-dimensional markets it becomes computationally challenging. Here, we choose the second route, which relies on the affine equivariance of the elliptical distributions.

First, we produce an auxiliary set of scenarios:

\[ \left( \hat{p}_j, \hat{\mathbf{x}}_j \right), \quad j = 1, \ldots, J/2 \]

from the distribution \( \mathcal{N}(\mathbf{0}, \mathbf{S}) \). Then we complement these scenarios with their opposite:

\[ \left( \hat{p}_j, \hat{\mathbf{x}}_j \right) = \begin{cases} \left( \hat{p}_j, \mathbf{x}_j \right) & \text{if } j \leq J/2 \\ \left( \hat{p}_{j-J/2}, -\mathbf{x}_j \right) & \text{if } j > J/2 \end{cases} \]

These antithetic variables still represent the distribution \( \mathcal{N}(\mathbf{0}, \mathbf{S}) \), but they are more efficient (see Boyle, Broadie & Glasserman, 1999) and they satisfy the zero-mean condition \( \hat{\mathbf{m}} = \mathbf{0} \).

To perform risk and portfolio management, we must represent the distribution of the risk factors that affect the market. The most flexible approach is in terms of scenarios and their probabilities, which includes historical scenarios, pure Monte Carlo and importance sampling (see Glasserman, 2004).

The same problem has been tackled by, among others, Wenderburn (1975), Cheng (1985), Li (1992) and Alexander, Lederburn (1975), Cheng (1985), Li (1992) and Alexander, Leder-
Next we perform its Schur decomposition:

\[ H = U T U^\ast \]  

(9)

where \( U T \) is upper triangular with the eigenvalues of \( H \) on the diagonal sorted in such a way that the first \( J \) have positive real part; the terms in this decomposition are similar in nature to principal components and are calculated by standard software packages (see, for example, Anderson et al., 1999). Then the solution of the Riccati equation (7) reads:

\[ B = U_{\leq J}U_{\leq J}^\ast \]  

(10)

where \( U_{\leq J} \) is the upper left \( J \times J \) block of \( U \) and \( U_{\leq J} \) is the lower left \( N \times J \) block of \( U \).

With the solution (10) we can perform the affine transformation (6) and finally generate the desired scenarios:

\[ x_j = m + y_j, \quad j = 1, \ldots, J \]  

(11)

which satisfy (2), see figure 1, where, as in Meucci (2005), we represent the first two moments of a distribution in terms of an ellipsoid.

Note that the steps (4)–(11) only require a few fractions of a second to run even for large problems. Refer to www.symmys.com ⇒ Teaching ⇒ MATLAB for a fully functional implementation.

Applications

To illustrate the ubiquitous nature of normal simulations, here we apply our methodology to calculate the value-at-risk and its decomposition in a book of \( I \) plain vanilla call options (see also Meucci, 2008). We denote current time as \( T \) and finally generate the desired scenarios:

\[ P(T \leq t) = P \left( X_{t,T}, X_{T,t}, T \right) \]  

(12)

where \( (X_{T,t}, X_{t,T}) \) are risk factors and \( T \) represents currently available information. Indeed, consider the Black-Scholes pricing formula:

\[ C_B \left( y, \sigma, K, T \right) = \Phi \left( d_1 \right) - Ke^{-rt} \Phi \left( d_2 \right) \]  

(13)

where \( \Phi \) is the cumulative distribution function of the standard normal distribution and:

\[ d_1 = \frac{-\ln \left( K / y \right) + \left( r + \sigma^2 / 2 \right)}{\sqrt{\sigma^2 t}} \]  

(14)

\[ d_2 = \frac{-\ln \left( K / y \right) + \left( r - \sigma^2 / 2 \right)}{\sqrt{\sigma^2 t}} \]  

(15)

Then:

\[ P(T \leq t) = C_B \left( y_T e^{X_{t,T}}, h \left( y_T e^{X_{t,T}}, \sigma, e^{X_{t,T}}, K, T - \tau \right), K, T - \tau, \tau \right) \]  

(16)

In this expression \( y_T \) is the current value and \( X_{t,T} = \ln \left( y_T / y_i \right) \) is the log-change of the underlying; \( y_i \) is the current value and \( X_{t,T} = \ln \left( \sigma \sigma_i / \sigma \right) \) is the log-change in \( (T - \tau) \)-expiry, at-the-money implied volatility; and \( h \) is a skew/smile map:

\[ h \left( y, \sigma, K, T \right) = \sigma + \alpha \frac{\ln \left( y / K \right)}{\sqrt{T}} + \beta \left( \ln \left( y / K \right) \right)^2 \]  

(17)

for coefficients \( \alpha \) and \( \beta \), which depend on the underlying and are fitted empirically, as in Malz (1997). Clearly, (16) is in the format (12).

Consider a portfolio represented by the vector \( w \), whose generic \( i \)th entry is the number of contracts in the respective call. The profit and loss then reads:

\[ H_w = \sum_{i=1}^{I} w_i \left( P \left( X_{t,T}^{(i)}, X_{T,t}^{(i)}, T \right) - p_{y_i} \right) \]  

(18)

where \( p_{y_i} \) denotes the currently traded price of the \( i \)th call. In order to calculate the VAR, we need the distribution of the profit and loss (18). To obtain the latter, we need the joint distribution of all the sources of risk \( (X_{t,T}^{(i)}, X_{T,t}^{(i)}) \) in the portfolio. We realise that the sources of risk are approximately invariants – their joint distribution is independent and identical across time, and thus it does not depend on the specific time. We model this joint distribution with a multivariate normal copula with non-parametric marginal:

\[ \begin{bmatrix} X_{t,T}^{(i)} \\ X_{T,t}^{(i)} \\ \vdots \\ X_{t,T}^{(I)} \\ X_{T,t}^{(I)} \end{bmatrix} \sim \begin{bmatrix} \tilde{F}_{X_t^{(i)}} \Phi \left( Z_{t,i}^{(i)} \right) \\ \tilde{F}_{X_T^{(i)}} \Phi \left( Z_T^{(i)} \right) \\ \vdots \\ \tilde{F}_{X_t^{(I)}} \Phi \left( Z_{t,I}^{(I)} \right) \\ \tilde{F}_{X_T^{(I)}} \Phi \left( Z_T^{(I)} \right) \end{bmatrix} \]  

(19)

In this expression:

\[ Z = N \left( 0, \hat{C} \right) \]  

(20)
for a suitably estimated correlation matrix \( \mathbf{C} \), and \( \hat{\mathbf{F}} \) denotes a suitable estimate of the marginal distribution of \( X \). In particular, we estimate these marginals by a non-parametric kernel smoothing of the historical data. Then we use the cumulative density functions \( \hat{F}_i \) to invert (19) for each entry in our joint time series of log price changes \( X_{n,t} \) and the log volatility changes \( X_{n,\sigma^2} \):

\[
Z_{n,t}^{(i)} = \Phi^{-1}(\hat{F}_{n,t}^{-1}(X_{n,t}^{(i)})), \quad t = 1, \ldots, T \tag{21}
\]

where this simplified notation applies to both \( y \) and \( \sigma \). Next, we fit the correlation (20) to the time series (21).

Now we can simulate the profit and loss distribution (18). First, we use our recipe to draw Monte Carlo scenarios from (20) in such a way that the sample mean is zero and the sample covariance exactly matches the estimated correlation matrix \( \mathbf{C} \). More precisely, we generate a \( J \times 2I \) panel \( Y \) of \( J \) normal simulations with sample mean a \( 2I \) vector of zeros and sample covariance the \( 2I \times 2I \) identity matrix, and map it into a \( J \times 2I \) panel \( \mathbf{Z} \) distributed as \( \mathbf{w} \) using the Cholesky decomposition of \( \mathbf{C} \). Then we map those simulations into factor realisations using (19). Since the expression of the inverse cumulative distribution function is not available analytically, we perform a linear interpolation of the cumulative density function, as in Hallerbach (2003). Then we map those simulations to the Cholesky decomposition of \( \mathbf{C} \), thereby generating a \( J \times I \) panel \( \mathcal{H} \) of joint profit and loss scenarios for the \( J \) assets.

To calculate the VAR and its contributions from the different securities in the portfolio, first we express the former in terms of the latter:

\[
\text{VAR} = \sum_{j=1}^{J} \frac{\partial \text{VAR}}{\partial w_j} \tag{22}
\]

where the partial derivatives that appear in (22) can be expressed conveniently as in Hallerbach (2003) and Gourieroux, Laurent & Scaillet (2000):

\[
\frac{\partial \text{VAR}}{\partial w} = \mathbf{P} - \mathbb{E} \left\{ \mathbf{H} \mathbf{w} = -\text{VAR} \right\} \tag{23}
\]

In this expression, \( \mathcal{H} \) is a \( J \times I \) panel whose generic \( i \)-th column is the \( i \)-th column of the options profit and loss panel \( \mathcal{H} \), sorted as the order statistics of the \( J \)-dimensional vector of the portfolio losses \( -\mathbf{H} \mathbf{w} \), and \( k \) is a Gaussian smoothing kernel that peaks around the rescaled confidence level \( c/p \). Finally, (22) yields the contributions from each option as well as the total VAR. The total VAR number then follows from (22).

In figure 3, we report the total VAR in our example of a portfolio long-short 20 options, as well as its decomposition in terms of the contributions from each call.

Risk managers can now proceed to stress test the correlation \( \mathbf{C} \) using the Cholesky decomposition of the new stress-test matrix and the \( J \times 2I \) panel \( \mathbf{Y} \) of uncorrelated standard normal simulations in the above process. Then they can analyse the impact of the stress test on a risk report such as figure 3, confident that the stress-test assumptions will be faithfully reflected in the simulations.

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